

## Chapter 4: Definite Integration

### EXERCISE 4.1 [PAGE 156]

#### Exercise 4.1 | Q 1 | Page 156

Evaluate the following integrals as limit of a sum :  $\int_1^3 (3x - 4) \cdot dx$

#### SOLUTION

Let  $f(x) = 3x - 4$ , for  $1 \leq x \leq 3$

Divide the closed interval  $[1, 3]$  into  $n$  subintervals each of length  $h$  at the points

$$1, 1 + h, 1 + 2h, 1 + rh, \dots, 1 + nh = 3$$

$$\therefore nh = 2$$

$$\therefore h = \frac{2}{n} \text{ and as } n \rightarrow \infty, h \rightarrow 0$$

Here,  $a = 1$

$$\therefore f(a + rh) = f(1 + rh) = 3(1 + rh) - 4 = 3rh - 1$$

$$\therefore \int_a^b f(x) \cdot dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n f(a + rh) \cdot h$$

$$\therefore \int_1^3 (3x - 4) \cdot dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n f(3rh - 1) \cdot h$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left( 3r \cdot \frac{2}{n} - 1 \right) \cdot \frac{2}{n} \quad \dots [\because h = \frac{2}{n}]$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left( \frac{12}{n^2} - \frac{2}{n} \right)$$

$$= \lim_{n \rightarrow \infty} \left[ \frac{12}{n^2} \sum_{r=1}^n r - \frac{2}{n} \sum_{r=1}^n 1 \right]$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \left[ \frac{12}{n^2} \frac{n(n+1)}{2} \frac{2}{n} \cdot n \right] \\
&= \lim_{n \rightarrow \infty} \left[ 6 \left( \frac{n+1}{n} \right) - 2 \right] \\
&= \lim_{n \rightarrow \infty} \left[ 6 \left( 1 + \frac{1}{n} \right) - 2 \right] \\
&= 6(1 + 0) - 2 \quad \dots [\because \lim_{n \rightarrow \infty} \frac{1}{n} = 0] \\
&= 4.
\end{aligned}$$

Exercise 4.1 | Q 2 | Page 156

Evaluate the following integrals as limit of a sum :  $\int_0^4 x^2 \cdot dx$

**SOLUTION**

Let  $f(x) = x^2$ , for  $0 \leq x \leq 4$

Divide the closed interval  $[0, 4]$  into  $n$  subintervals each of length  $h$  at the points  $0, 0 + h, 0 + 2h, \dots, 0 + rh, \dots, 0 + nh = 4$

i.e.  $0, h, 2h, \dots, rh, \dots, nh = 4$

$$\therefore h = \frac{4}{n} \text{ as } n \rightarrow \infty, h \rightarrow 0$$

Here,  $a = 0$

$$\therefore f(a + rh) = f(0 + rh) = f(rh) = r^2 h^2$$

$$\therefore \int_a^b f(x) \cdot dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n f(a + rh) \cdot h$$

$$\therefore \int_0^4 x^2 \cdot dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n r^2 h^2 \cdot h$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n r^2 \frac{+64}{n^3} \dots [\because h = \frac{4}{n}]$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \left[ \frac{64}{n^3} \sum_{r=1}^n r^2 \right] \\
&= \lim_{n \rightarrow \infty} \left[ \frac{64}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \right] \\
&= \lim_{n \rightarrow \infty} \left[ \frac{64}{6} \left( \frac{n+1}{n} \right) \left( \frac{2n+1}{n} \right) \right] \\
&= \lim_{n \rightarrow \infty} \left[ \frac{64}{6} \left( 1 + \frac{1}{n} \right) \left( 2 + \frac{1}{n} \right) \right] \\
&= \frac{64}{6} (1+0)(2+0) \dots \left[ \because \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \right] \\
&= \frac{64}{3}.
\end{aligned}$$

Exercise 4.1 | Q 3 | Page 156

Evaluate the following integrals as limit of a sum :  $\int_0^2 e^x \cdot dx$

**SOLUTION**

Let  $f(x) = e^x$ , for  $0 \leq x \leq 2$

Divide the closed interval  $[0, 2]$  into  $n$  equal subintervals each of length  $h$  at the points  $0, 0+h, 0+2h, \dots, 0+rh, \dots, 0+nh=2$

i.e.  $0, h, 2h, \dots, rh, \dots, nh=2$

$$\therefore h = \frac{2}{n} \text{ and } n \rightarrow \infty, h \rightarrow 0$$

Here,  $a = 0$

$$\therefore f(a+rh) = f(0+rh) = f(rh) = e^{rh}$$

$$\therefore \int_a^b f(x) \cdot dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n f(a+rh) \cdot h$$

$$\therefore \int_0^2 e^x \cdot dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n e^{rh} \cdot h$$

$$= \lim_{h \rightarrow 0} \left[ h \sum_{r=1}^n e^{rh} \right] \dots [as n \rightarrow \infty, h \rightarrow 0]$$

$$\begin{aligned} \text{Now, } \sum_{r=1}^n e^{rh} &= e^h + e^{2h} + \dots + e^{nh} \\ &= \frac{e^h [(e^h)^n - 1]}{e^h - 1} \\ &= \frac{e^h [e^{nh} - 1]}{e^h - 1} \\ &= \frac{e^h \cdot (e^2 - 1)}{e^h - 1} \dots \left[ \because h = \frac{2}{n} \therefore nh = 2 \right] \end{aligned}$$

$$\begin{aligned} &= (e^2 - 1) \frac{e^h}{e^h - 1} \\ \therefore \int_0^2 e^x \cdot dx &= \lim_{h \rightarrow 0} \frac{h(e^2 - 1)e^h}{e^h - 1} \\ &= (e^2 - 1) \lim_{h \rightarrow 0} \frac{e^h}{\left(\frac{e^h - 1}{h}\right)} \\ &= (e^2 - 1) \frac{\lim_{h \rightarrow 0} e^h}{\lim_{h \rightarrow 0} \left(\frac{e^h - 1}{h}\right)} \\ &= (e^2 - 1) \cdot \frac{e^0}{1} \dots \left[ \because \lim_{h \rightarrow 0} \frac{e^{h-1}}{h} = 1 \right] \\ &= (e^2 - 1). \end{aligned}$$

Exercise 4.1 | Q 4 | Page 156

Evaluate the following integrals as limit of a sum :  $\int_0^2 (3x^2 - 1) \cdot dx$

**SOLUTION**

Let  $f(x) = 3x^2 - 1$ , for  $0 \leq x \leq 2$ .

Divide the closed interval  $[0, 2]$  into  $n$  subintervals each of length  $h$  at the points.

$0, 0 + h, 0 + 2h, \dots, 0 + rh, \dots, 0 + nh = 2$

i.e.  $0, h, 2h, \dots, rh, \dots, nh = 2$

$$\therefore h = \frac{2}{n} \text{ and as } n \rightarrow \infty, h \rightarrow 0$$

Here,  $a = 0$

$$\therefore f(a + rh) = f(0 + rh) = f(rh) = 3(rh)^2 - 1 = 3r^2h^2 - 1$$

$$\begin{aligned} \therefore \int_a^b f(x) \cdot dx &= \lim_{n \rightarrow \infty} \sum_{r=1}^n f(a + rh) \cdot h \\ &= \int_0^2 (3x^2 - 1) \cdot dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n (3r^2h^2 - 1) \cdot h \end{aligned}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left( 3r^2 \times \frac{4}{n^2} - 1 \right) \cdot \frac{2}{n} \dots [\because h = \frac{2}{n}]$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left( \frac{24r^2}{n^3} - \frac{2}{n} \right)$$

$$= \lim_{n \rightarrow \infty} \left[ \frac{24}{n^3} \sum_{r=1}^n r^2 - \frac{2}{n} \sum_{r=1}^n 1 \right]$$

$$= \lim_{n \rightarrow \infty} \left[ \frac{24}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{2}{n} \cdot n \right]$$

$$= \lim_{n \rightarrow \infty} \left[ 4 \cdot \left( \frac{n+1}{n} \right) \left( \frac{2n+1}{n} \right) - 2 \right]$$

$$= \lim_{n \rightarrow \infty} \left[ 4 \left( 1 + \frac{1}{n} \right) \left( 2 + \frac{1}{n} \right) - 2 \right]$$

$$= 4(1+0)(2+0) - 2 \dots [\because \lim_{n \rightarrow \infty} \frac{1}{n} = 0]$$

$$= 8 - 2 = 6.$$

Evaluate the following integrals as limit of a sum :  $\int_1^3 x^3 \cdot dx$

**SOLUTION**

Let  $f(x) = x^3$ , for  $1 \leq x \leq 3$ .

Divide the closed interval  $[1, 3]$  into  $n$  equal subintervals each of length  $h$  at the points  $1, 1 + h, 1 + 2h, \dots, 1 + rh, \dots, 1 + nh = 3$

$$\therefore nh = 2$$

$$\therefore h = \frac{2}{n} \text{ and as } n \rightarrow \infty, h \rightarrow 0.$$

Here,  $a = 1$ .

$$\therefore f(a + rh) = f(1 + rh) = (1 + rh)^3$$

$$= 1 + 3rh + 3r^2h^2 + r^3h^3$$

$$\therefore \int_a^b f(x) \cdot dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n f(a + rh) \cdot h$$

$$\therefore \int_1^3 x^3 \cdot dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n (1 + 3rh + 3r^2h^2 + r^3h^3) \cdot h$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n (h + 3rh^2 + 3r^2h^3 + r^3h^4)$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left[ \frac{2}{n} + 3r\left(\frac{2}{n}\right)^2 + 3r^2\left(\frac{2}{n}\right)^3 + r^3\left(\frac{2}{n}\right)^4 \right] \dots \left[ \because h = \frac{2}{n} \right]$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left[ \frac{2}{n} + \frac{12r}{n^2} + \frac{24r^2}{n^3} + \frac{16r^3}{n^4} \right]$$



$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \left[ \frac{2}{n} \sum_{r=1}^n 1 + \frac{12}{n^2} \sum_{r=1}^n r + \frac{24}{n^3} \sum_{r=1}^n r^2 + \frac{16}{n^4} \sum_{r=1}^n r^3 \right] \\
&= \lim_{n \rightarrow \infty} \left[ \frac{2}{n} \cdot n + \frac{12}{n^2} \cdot \frac{n(n+1)}{2} + \frac{24}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} \right] \\
&= \lim_{n \rightarrow \infty} \left[ 2 + 6\left(\frac{n+1}{n}\right) + 4\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) + 4\left(\frac{n+1}{n}\right)^2 \right] \\
&= \lim_{n \rightarrow \infty} \left[ 2 + 6\left(1 + \frac{1}{n}\right) + 4\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 4\left(1 + \frac{1}{n}\right)^2 \right] \\
&= [2 + 6(1 + 0) + 4(1 + 0)(2 + 0) + 4(1 + 0)^2] \dots [\because \lim_{n \rightarrow \infty} \frac{1}{n} = 0] \\
&= 2 + 6 + 8 + 4 \\
&= 20.
\end{aligned}$$

#### EXERCISE 4.2 [PAGES 171 - 172]

Exercise 4.2 | Q 1.01 | Page 171

Evaluate :  $\int_1^9 \frac{x+1}{\sqrt{x}} \cdot dx$

**SOLUTION**

$$\begin{aligned}
\int_1^9 \frac{x+1}{\sqrt{x}} \cdot dx &= \int_1^9 \left( \frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} \right) \cdot dx \\
&= \int_1^9 x^{\frac{1}{2}} \cdot dx + \int_1^9 x^{-\frac{1}{2}} \cdot dx \\
&= \left[ \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^9 + \left[ \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^9 \\
&= \frac{2}{3} \left[ 9^{\frac{3}{2}} - 1^{\frac{3}{2}} \right] + 2 \left[ 9^{\frac{1}{2}} - 1^{\frac{1}{2}} \right] \\
&= \frac{2}{3} \left[ (3^2)^{\frac{3}{2}} - 1 \right] + 2[3 - 1]
\end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{3}[27 - 1] + 4 \\
 &= \frac{52}{3} + 4 \\
 &= \frac{64}{3}.
 \end{aligned}$$

Exercise 4.2 | Q 1.02 | Page 171

Evaluate :  $\int_2^3 \frac{1}{x^2 + 5x + 6} \cdot dx$

**SOLUTION**

$$\begin{aligned}
 &\int_2^3 \frac{1}{x^2 + 5x + 6} \cdot dx \\
 &= \int_2^3 \frac{1}{(x + 2)(x + 3)} \cdot dx \\
 &= \int_2^3 \frac{(x + 3) - (x + 2)}{(x + 2)(x + 3)} \cdot dx \\
 &= \int_2^3 \left[ \frac{1}{x + 2} - \frac{1}{x + 3} \right] \cdot dx \\
 &= [\log(x + 2) - \log(x + 3)]_2^3 \\
 &= \left[ \log \left| \frac{x + 2}{x + 3} \right| \right]_2^3 \\
 &= \log \left( \frac{3 + 2}{3 + 3} \right) - \log \left( \frac{2 + 2}{2 + 3} \right) \\
 &= \log \frac{5}{6} - \log \frac{4}{5} \\
 &= \log \left( \frac{5}{6} \times \frac{5}{4} \right)
 \end{aligned}$$



$$= \log\left(\frac{25}{24}\right).$$

Exercise 4.2 | Q 1.03 | Page 171

Evaluate :  $\int_0^{\frac{\pi}{4}} \cot^2 x \cdot dx$

**SOLUTION**

$$\begin{aligned} & \int_0^{\frac{\pi}{4}} \cot^2 x \cdot dx \\ &= \int_0^{\frac{\pi}{4}} (\operatorname{cosec}^2 x - 1) \cdot dx \\ &= \int_0^{\frac{\pi}{4}} \operatorname{cosec}^2 x \cdot dx - \int_0^{\frac{\pi}{4}} 1 \cdot dx \\ &= [-\cot x]_0^{\frac{\pi}{4}} - [x]_0^{\frac{\pi}{4}} \\ &= \left[ \left( -\frac{\cot \pi}{4} \right) - (-\cot 0) \right] - \left[ \frac{\pi}{4} - 0 \right] \\ &= -1 + \cot 0 - \frac{\pi}{4}. \end{aligned}$$

The integral does not exist since  $\cot 0$  is not defined.

Exercise 4.2 | Q 1.04 | Page 171

Evaluate :  $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1 - \sin x} \cdot dx$

**SOLUTION**

$$\begin{aligned}& \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1 - \sin x} \cdot dx \\&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} \cdot dx \\&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1 + \sin x}{1 + \sin x} \cdot dx \\&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1 + \sin x}{\cos^2 x} \cdot dx \\&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left( \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \right) \cdot dx \\&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2 x + \sec x \tan x) \cdot dx \\&= [\tan x + \sec x]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\&= \left( \tan \frac{\pi}{4} + \sec \frac{\pi}{4} \right) - \left[ \tan \left( -\frac{\pi}{4} \right) + \sec \left( -\frac{\pi}{4} \right) \right] \\&= \left( 1 + \sqrt{2} \right) - \left( -\tan \frac{\pi}{4} + \sec \frac{\pi}{4} \right) \\&= \left( 1 + \sqrt{2} \right) - \left( -1 + \sqrt{2} \right) \\&= 1 + \sqrt{2} + 1 - \sqrt{2} \\&= 2.\end{aligned}$$

Exercise 4.2 | Q 1.05 | Page 171

Evaluate :  $\int_3^5 \frac{1}{\sqrt{2x+3} - \sqrt{2x-3}} \cdot dx$



**SOLUTION**

$$\begin{aligned}& \int_3^5 \frac{1}{\sqrt{2x+3} - \sqrt{2x-3}} \cdot dx \\&= \int_3^5 \frac{1}{\sqrt{2x+3} - \sqrt{2x-3}} \times \frac{\sqrt{2x+3} + \sqrt{2x-3}}{\sqrt{2x+3} + \sqrt{2x-3}} \cdot dx \\&= \int_3^5 \frac{\sqrt{2x+3} + \sqrt{2x-3}}{(2x+3) - (2x-3)} \cdot dx \\&= \frac{1}{6} \int_3^5 (2+3)^{\frac{1}{2}} \cdot dx + \frac{1}{6} \int_3^5 (2x-3)^{\frac{1}{2}} \cdot dx \\&= \frac{1}{6} \left[ \frac{2x+3^{\frac{3}{2}}}{2(\frac{3}{2})} \right]_3^5 + \frac{1}{6} \left[ \frac{(2x-3)^{\frac{3}{2}}}{2(\frac{3}{2})} \right]_3^5 \\&= \frac{1}{18} \left[ (10+3)^{\frac{3}{2}} - (6+3)^{\frac{3}{2}} \right] + \frac{1}{18} \left[ (10-3)^{\frac{3}{2}} - (6-3)^{\frac{3}{2}} \right] \\&= \frac{1}{18} \left[ 13\sqrt{13} - 9\sqrt{9} \right] + \frac{1}{18} \left[ 7\sqrt{7} - 3\sqrt{3} \right] \\&= \frac{1}{18} \left( 13\sqrt{13} - 27 + 7\sqrt{7} - 3\sqrt{3} \right) \\&= \frac{1}{18} \left( 13\sqrt{13} + 7\sqrt{7} - 3\sqrt{3} - 27 \right).\end{aligned}$$

Exercise 4.2 | Q 1.06 | Page 171

Evaluate :  $\int_0^1 \frac{x^2 - 2}{x^2 + 1} \cdot dx$



**SOLUTION**

$$\begin{aligned}& \int_0^1 \frac{x^2 - 2}{x^2 + 1} \cdot dx \\&= \int_0^1 \frac{(x^2 + 1) - 3}{x^2 + 1} \cdot dx \\&= \int_0^1 \left( 1 - \frac{3}{x^2 + 1} \right) \cdot dx \\&= [x - 3 \tan^{-1} x]^1 \\&= (1 - 3 \tan^{-1} 1) - (0 - 3 \tan^{-1} 0) \\&= 1 - 3 \left( \frac{\pi}{4} \right) - 0 \\&= 1 - \frac{3\pi}{4}.\end{aligned}$$

Exercise 4.2 | Q 1.07 | Page 171

Evaluate :  $\int_0^{\frac{\pi}{4}} \sin 4x \sin 3x \cdot dx$

**SOLUTION**

$$\begin{aligned}& \int_0^{\frac{\pi}{4}} \sin 4x \sin 3x \cdot dx \\&= \frac{1}{2} \int_0^{\frac{\pi}{4}} 2 \sin 4x \sin 3x \cdot dx \\&= \frac{1}{2} \int_0^{\frac{\pi}{4}} [\cos(4x - 3x) - \cos(4x + 3x)] \cdot dx \\&= \frac{1}{2} \int_0^{\frac{\pi}{4}} \cos x \cdot dx - \frac{1}{2} \int_0^{\frac{\pi}{4}} \cos 7x \cdot dx\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} [\sin x]_0^{\frac{\pi}{4}} - \frac{1}{2} \left[ \frac{\sin 7x}{7} \right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{2} \left[ \sin \frac{\pi}{4} - \sin 0 \right] - \frac{1}{14} \left[ \sin \frac{7\pi}{4} - \sin 0 \right] \\
&= \frac{1}{2} \left[ \frac{1}{\sqrt{2}} - 0 \right] - \frac{1}{14} \left[ \sin \left( 2\pi - \frac{\pi}{4} \right) - 0 \right] \\
&= \frac{1}{2\sqrt{2}} - \frac{1}{14} \left( -\sin \frac{\pi}{4} \right) \\
&= \frac{1}{2\sqrt{2}} + \frac{1}{14\sqrt{2}} \\
&= \frac{7+1}{14\sqrt{2}} \\
&= \frac{4}{7\sqrt{2}}.
\end{aligned}$$

Exercise 4.2 | Q 1.08 | Page 171

Evaluate :  $\int_0^{\frac{\pi}{4}} \sqrt{1 + \sin 2x} \cdot dx$

**SOLUTION**

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} \sqrt{1 + \sin 2x} \cdot dx \\
&= \int_0^{\frac{\pi}{4}} \sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x} \cdot dx \\
&= \int_0^{\frac{\pi}{4}} \sqrt{(\sin x + \cos x)^2} \cdot dx \\
&= \int_0^{\frac{\pi}{4}} (\sin x + \cos x) \cdot dx
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{4}} \sin x \cdot dx + \int_0^{\frac{\pi}{4}} \cos x \cdot dx \\
&= [-\cos x]_0^{\frac{\pi}{4}} + [\sin x]_0^{\frac{\pi}{4}} \\
&= \left[ -\cos \frac{\pi}{4} - (-\cos 0) \right] + \left[ \sin \frac{\pi}{4} - \sin 0 \right] \\
&= -\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} - 0 \\
&= 1.
\end{aligned}$$

Exercise 4.2 | Q 1.09 | Page 171

Evaluate :  $\int_0^{\frac{\pi}{4}} \sin^4 x \cdot dx$

**SOLUTION**

$$\begin{aligned}
&\text{Consider } \sin^4 x = (\sin^2 x)^2 \\
&= \left( \frac{1 - \cos 2x}{2} \right)^2 \\
&= \frac{1}{4} [1 - 2 \cos 2x + \cos^2 2x] \\
&= \frac{1}{4} \left[ 1 - 2 \cos 2x + \frac{1 + \cos 4x}{2} \right] \\
&= \frac{1}{4} \left[ \frac{3}{2} - 2 \cos 2x + \frac{1}{2} \cos 4x \right] \\
&\therefore \int_0^{\frac{\pi}{4}} \sin^4 x \cdot dx \\
&= \frac{1}{4} \int_0^{\frac{\pi}{4}} \left[ \frac{3}{2} - 2 \cos 2x + \frac{1}{2} \cos 4x \right] \cdot dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{3}{8} \int_0^{\frac{\pi}{4}} 1 \cdot dx - \frac{1}{2} \int_0^{\frac{\pi}{4}} \cos 2x \cdot dx + \frac{1}{8} \int_0^{\frac{\pi}{4}} \cos 4x \cdot dx \\
&= \frac{3}{8} [x]_0^{\frac{\pi}{4}} - \frac{1}{2} \left[ \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{4}} + \frac{1}{8} \left[ \frac{\sin 4x}{4} \right]_0^{\frac{\pi}{4}} \\
&= \frac{3}{8} \left[ \frac{\pi}{4} - 0 \right] - \frac{1}{4} \left[ \sin \frac{\pi}{2} - \sin 0 \right] + \frac{1}{32} [\sin \pi - \sin 0] \\
&= \frac{3\pi}{32} - \frac{1}{4} [1 - 0] + \frac{1}{32} (0 - 0) \\
&= \frac{3\pi}{32} - \frac{1}{4} \\
&= \frac{3\pi - 8}{32}.
\end{aligned}$$

Exercise 4.2 | Q 1.1 | Page 171

Evaluate :  $\int_{-4}^2 \frac{1}{x^2 + 4x + 13} \cdot dx$

**SOLUTION**

$$\begin{aligned}
&\int_{-4}^2 \frac{1}{x^2 + 4x + 13} \cdot dx \\
&= \int_{-4}^2 \frac{1}{x^2 + 4x + 4 + 9} \cdot dx \\
&= \int_{-4}^2 \frac{1}{(x + 2)^2 + 3^2} \cdot dx \\
&= \left[ \frac{1}{3} \tan^{-1} \left( \frac{x + 2}{3} \right) \right]_{-4}^2 \\
&= \frac{1}{3} \tan^{-1} \left( \frac{2 + 2}{3} \right) - \frac{1}{3} \tan^{-1} \left( \frac{-4 + 2}{3} \right)
\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{3} \tan^{-1} \left( \frac{4}{3} \right) - \frac{1}{3} \tan^{-1} \left( -\frac{2}{3} \right) \\
 &= \frac{1}{3} \left[ \tan^{-1} \frac{4}{3} + \tan^{-1} \frac{2}{3} \right]. \quad \dots [\because \tan^{-1} (-x) = -\tan^{-1} x]
 \end{aligned}$$

Exercise 4.2 | Q 1.11 | Page 171

Evaluate :  $\int_0^4 \frac{1}{\sqrt{4x - x^2}} \cdot dx$

**SOLUTION**

$$\begin{aligned}
 &\int_0^4 \frac{1}{\sqrt{4x - x^2}} \cdot dx \\
 &= \int_0^4 \frac{1}{\sqrt{4 - (x^2 - 4x + 4)}} \cdot dx \\
 &= \int_0^4 \frac{1}{\sqrt{2^2 - (x - 2)^2}} \cdot dx \\
 &= \left[ \sin^{-1} \left( \frac{x - 2}{2} \right) \right]_0^4 \\
 &= \sin^{-1} \left( \frac{4 - 2}{2} \right) - \sin^{-1} \left( \frac{0 - 2}{2} \right) \\
 &= \sin^{-1} 1 - \sin^{-1} (-1) \\
 &= 2 \sin^{-1} 1 \quad \dots [\because \sin^{-1} (-x) = -\sin^{-1} x] \\
 &= 2 \left( \frac{\pi}{2} \right) \\
 &= \pi.
 \end{aligned}$$



Exercise 4.2 | Q 1.12 | Page 171

Evaluate :  $\int_0^1 \frac{1}{\sqrt{3+2x-x^2}} \cdot dx$

**SOLUTION**

$$\begin{aligned} & \int_0^1 \frac{1}{\sqrt{3+2x-x^2}} \cdot dx \\ &= \int_0^1 \frac{1}{\sqrt{3-(x^2-2x+1)+1}} \cdot dx \\ &= \int_0^1 \frac{1}{\sqrt{(2)^2-(x-1)^2}} \cdot dx \\ &= \left[ \sin^{-1} \left( \frac{x-1}{2} \right) \right]_0^1 \\ &= \sin^{-1}(0) - \sin^{-1} \left( -\frac{1}{2} \right) \\ &= 0 - \sin^{-1} \left( -\sin \frac{\pi}{6} \right) \\ &= -\sin^{-1} \left[ \sin \left( -\frac{\pi}{6} \right) \right] \\ &= - \left( -\frac{\pi}{6} \right) \\ &= \frac{\pi}{6}. \end{aligned}$$

Exercise 4.2 | Q 1.13 | Page 171

Evaluate :  $\int_0^{\frac{\pi}{2}} x \sin x \cdot dx$

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**SOLUTION**

$$\begin{aligned}& \int_0^{\frac{\pi}{2}} x \sin x \cdot dx \\&= \left[ x \int \sin x \cdot dx \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left[ \frac{d}{dx}(x) \int \sin x \cdot dx \right] \cdot dx \\&= [x(-\cos x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 1 \cdot (-\cos x) \cdot dx \\&= -[x \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \cdot dx \\&= -\left[ \frac{\pi}{2} \cos \frac{\pi}{2} - 0 \right] + [\sin x]_0^{\frac{\pi}{2}} \\&= 0 + \left( \sin \frac{\pi}{2} - \sin 0 \right) \\&= 1.\end{aligned}$$

Exercise 4.2 | Q 1.14 | Page 171

Evaluate :  $\int_0^1 x \tan^{-1} x \cdot dx$

**SOLUTION**

$$\begin{aligned}\text{Let } I &= \int_0^1 x \tan^{-1} x \cdot dx \\&= \int_0^1 (\tan^{-1} x)(x) \cdot dx \\&= \left[ (\tan^{-1} x) \int x \cdot dx \right]_0^1 - \int_0^1 \left[ \frac{d}{dx}(\tan^{-1} x) \cdot \int x \cdot dx \right] \cdot dx \\&= \left[ \frac{x^2 \tan^{-1} x}{2} \right]_0^1 - \int_0^1 \frac{1}{1+x^2} \cdot \frac{x^2}{2} \cdot dx\end{aligned}$$

$$\begin{aligned}
&= \left( \frac{1^2 \tan^{-1} 1}{2} - 0 \right) - \frac{1}{2} \int_0^1 \frac{1+x^2-1}{1+x^2} \cdot dx \\
&= \frac{\frac{\pi}{4}}{2} - \frac{1}{2} \int_0^1 \left( 1 - \frac{1}{1+x^2} \right) \cdot dx \\
&= \frac{\pi}{8} - \frac{1}{2} [x - \tan^{-1}(x)]_0^1 \\
&= \frac{\pi}{8} - \frac{1}{2} [(1 - \tan^{-1} 1) - 0] \\
&= \frac{\pi}{8} - \frac{1}{2} \left( 1 - \frac{\pi}{4} \right) \\
&= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} \\
&= \frac{\pi}{4} - \frac{1}{2}.
\end{aligned}$$

Exercise 4.2 | Q 1.15 | Page 171

Evaluate :  $\int_0^{\infty} x e^{-x} \cdot dx$

**SOLUTION**

$$\begin{aligned}
&\int_0^{\infty} x e^{-x} \cdot dx \\
&= \left[ x \int e^{-x} \cdot dx \right]_0^{\infty} - \int_0^{\infty} \left[ \frac{d}{dx}(x) \int e^{-x} \cdot dx \right] \cdot dx \\
&= \left[ x \left( \frac{e^{-x}}{-1} \right) \right]_0^{\infty} - \int_0^{\infty} 1 \cdot \frac{e^{-x}}{(-1)} \cdot dx \\
&= \left[ -\frac{x}{e^x} \right]_0^{\infty} + \int_0^{\infty} e^{-x} \cdot dx \\
&= \left[ -\frac{x}{e^x} \right]_0^{\infty} + [-e^x]_0^{\infty}
\end{aligned}$$

$$\begin{aligned}
 &= [0 - (-0)] + [0 - (-1)] \\
 &= 1. \quad \dots [\because e^0 = 1, e^{-x} = 0, \text{ when } x = \infty]
 \end{aligned}$$

Exercise 4.2 | Q 2.01 | Page 172

Evaluate :  $\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}} \cdot dx$

**SOLUTION**

$$\begin{aligned}
 \text{Let } I &= \int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}} \cdot dx \\
 &= \int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1-x^2)\sqrt{1-x^2}} \cdot dx
 \end{aligned}$$

Put  $\sin^{-1} x = t$

$$\therefore \frac{1}{\sqrt{1-x^2}} \cdot dx = dt$$

Also,  $x = \sin t$

When  $x = \frac{1}{\sqrt{2}}, t = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

When  $x = 0, t = \sin^{-1}0 = 0$

$$\begin{aligned}
 \therefore I &= \int_0^{\frac{\pi}{4}} \frac{t}{1-\sin^2 t} \cdot dt \\
 &= \int_0^{\frac{\pi}{4}} \frac{t}{\cos^2 t} \cdot dt
 \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{4}} t \sec^2 t \cdot dt \\
&= \left[ t \int \sec^2 t \cdot dt \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \left[ \frac{d}{dt}(t) \int \sec^2 t \cdot dt \right] \cdot dt \\
&= [t \tan t]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 1 \cdot \tan t \cdot dt \\
&= \left[ \frac{\pi}{4} \tan \frac{\pi}{4} - 0 \right] - [\log |\sec t|]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{4} - \left[ \log \left( \sec \frac{\pi}{4} \right) - \log(\sec 0) \right] \\
&= \frac{\pi}{4} - [\log \sqrt{2} - \log 1] \\
&= \frac{\pi}{4} - \frac{1}{2} \log 2. \quad \dots [\because \log 1 = 0]
\end{aligned}$$

Exercise 4.2 | Q 2.02 | Page 172

Evaluate :  $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3 \tan^2 x + 4 \tan x + 1} \cdot dx$

**SOLUTION**

Let  $I = \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3 \tan^2 x + 4 \tan x + 1} \cdot dx$

Put  $\tan x = t$

$\therefore \sec^2 x \cdot dx = dt$

When  $x = 0$ ,  $t = \tan 0 = 0$

When  $x = \frac{\pi}{4}$ ,  $t = \tan \frac{\pi}{4} = 1$

$\therefore I = \int_0^1 \frac{dt}{3t^2 + 4t + 1}$

$$\begin{aligned}
&= \frac{1}{3} \int_0^1 \frac{dt}{t^2 + \frac{4}{3}t + \frac{1}{3}} \\
&= \frac{1}{3} \int_0^1 \frac{dt}{t^2 + \frac{4t}{3} + \frac{4}{9} - \frac{4}{9} + \frac{1}{3}} \\
&= \frac{dt}{\left(t + \frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2} \\
&= \frac{1}{3} \frac{1}{2\left(\frac{1}{3}\right)} \left[ \log \left| \frac{t + \frac{2}{3} - \frac{1}{3}}{t + \frac{2}{3} + \frac{1}{3}} \right| \right]_0^1 \\
&= \frac{1}{2} \left[ \log \left( \frac{1 + \frac{1}{3}}{1 + 1} \right) - \log \left( \frac{0 + \frac{1}{3}}{0 + 1} \right) \right] \\
&= \frac{1}{2} \left[ \log \left( \frac{2}{3} \right) - \log \left( \frac{1}{3} \right) \right] \\
&= \frac{1}{2} \log 2.
\end{aligned}$$


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Exercise 4.2 | Q 2.03 | Page 172

Evaluate :  $\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{\sin^4 x + \cos^4 x} \cdot dx$

**SOLUTION**

$$\begin{aligned}
\text{Let } I &= \int_0^{\frac{\pi}{4}} \frac{\sin 2x}{\sin^4 x + \cos^4 x} \cdot dx \\
&= \int_0^{\frac{\pi}{4}} \frac{2 \sin x \cos x}{\sin^4 x + \cos^4 x} \cdot dx
\end{aligned}$$

Dividing each term by  $\cos^4 x$ , we get

$$I = \int_0^{\frac{\pi}{4}} \frac{2 \frac{\sin x}{\cos x} \cdot \frac{1}{\cos^2 x}}{\frac{\sin^4 x}{\cos^4 x} + 1} \cdot dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{2 \tan x \sec^2 x}{(\tan^2)^2 + 1} \cdot dx$$

Put  $\tan^2 x = t$

$$\therefore 2 \tan x \sec^2 x \cdot dx = dt$$

When  $x = 0$ ,  $t = \tan^2 0 = 0$

When  $x = \frac{\pi}{4}$ ,  $t = \tan^2 \frac{\pi}{4} = 1$

$$\therefore I = \int_0^1 \frac{dt}{1 + t^2}$$

$$= [\tan^{-1} t]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}$$

Exercise 4.2 | Q 2.04 | Page 172

Evaluate :  $\int_0^{2\pi} \sqrt{\cos x} \sin^3 x \cdot dx$

**SOLUTION**

$$\begin{aligned}
 \text{Let } I &= \int_0^{2\pi} \sqrt{\cos x} \sin^3 x \cdot dx \\
 &= \int_0^{2\pi} \sqrt{\cos x} \sin^2 x \sin x \cdot dx \\
 &= \int_0^{2\pi} \sqrt{\cos x} (1 - \cos^2 x) \sin x \cdot dx
 \end{aligned}$$

Put  $\cos x = t$

$$\therefore -\sin x \cdot dx = dt$$

$$\therefore \sin x \cdot dx = -dt$$

$$\text{When } x = 0, t = \cos 0 = 1$$

$$\text{When } x = 2\pi, t = \cos 2\pi = 1$$

$$\therefore I = \int_1^1 \sqrt{t}(1 - t^2)(-dt) = 0. \quad \dots \left[ \because \int_a^a f(x) \cdot dx = 0 \right]$$

Exercise 4.2 | Q 2.05 | Page 172

$$\text{Evaluate : } \int_0^{\frac{\pi}{2}} \frac{1}{5 + 4 \cos x} \cdot dx$$

**SOLUTION**

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{1}{5 + 4 \cos x} \cdot dx$$

$$\text{Put } \tan\left(\frac{x}{2}\right) = t$$

$$\therefore x = 2 \tan^{-1} t$$

$$\therefore dx = \frac{2dt}{1 + t^2}$$

and \_\_\_\_\_



$$\cos x = \frac{1 - t^2}{1 + t^2}$$

$$\text{When } x = \frac{\pi}{2}, t = \tan\left(\frac{\pi}{2}\right) = 1$$

$$\text{When } x = 0, t = \tan 0 = 0$$

$$\begin{aligned} \therefore I &= \frac{\frac{2dt}{1+t^2}}{5 + 4\left(\frac{1-t^2}{1+t^2}\right)} \\ &= \int_0^1 \frac{2dt}{5(1+t^2) + 4(1-t^2)} \\ &= 2 \int_0^1 \frac{1}{t^2 + 9} \cdot dt \\ &= 2 \left[ \frac{1}{3} \tan^{-1} \frac{t}{3} \right]_0^1 \\ &= 2 \left[ \frac{1}{3} \tan^{-1} \frac{1}{3} - \frac{1}{3} \tan^{-1} 0 \right] \\ &= \frac{2}{3} \tan^{-1} \frac{1}{3} - \frac{2}{3} \times 0 \\ &= \frac{2}{3} \tan^{-1} \left( \frac{1}{3} \right). \end{aligned}$$

Exercise 4.2 | Q 2.06 | Page 172

$$\text{Evaluate : } \int_0^{\frac{\pi}{4}} \frac{\cos x}{4 - \sin^2 x} \cdot dx$$

**SOLUTION**

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \frac{\cos x}{4 - \sin^2 x} \cdot dx$$

Put  $\sin x = t$

$$\therefore \cos x \cdot dx = dt$$

$$\text{When } x = \frac{\pi}{4}, t = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\text{When } x = 0, t = \sin 0 = 0.$$

$$\begin{aligned} \therefore I &= \int_0^{\frac{1}{\sqrt{2}}} \frac{dt}{2^2 - t^2} \\ &= \left[ \frac{1}{2(2)} \log \left| \frac{2+t}{2-t} \right| \right]_0^{\frac{1}{\sqrt{2}}} \\ &= \frac{1}{4} \left[ \log \left( \frac{2 + \frac{1}{\sqrt{2}}}{2 - \frac{1}{\sqrt{2}}} \right) - \log \left( \frac{2+0}{2-0} \right) \right] \\ &= \frac{1}{4} \left[ \log \left( \frac{2\sqrt{2} + 1}{2\sqrt{2} - 1} \right) - \log 1 \right] \\ &= \frac{1}{4} \log \left( \frac{2\sqrt{2} + 1}{2\sqrt{2} - 1} \right). \quad \dots [\because \log 1 = 0] \end{aligned}$$

Exercise 4.2 | Q 2.07 | Page 172

$$\text{Evaluate : } \int_0^{\frac{\pi}{2}} \frac{\cos x}{(1 + \sin x)(2 + \sin x)} \cdot dx$$

**SOLUTION**

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{(1 + \sin x)(2 + \sin x)} \cdot dx$$

Put  $\sin x = t$

$$\therefore \cos x \cdot dx = dt$$

$$\text{When } x = \frac{\pi}{2}, t = \sin \frac{\pi}{2} = 1$$

$$\text{When } x = 0, t = \sin 0 = 0$$

$$\begin{aligned}\therefore I &= \int_0^1 \frac{dt}{(1+t)(2+t)} \\&= \int_0^1 \frac{(2+t) - (1+t)}{(1+t)(2+t)} \cdot dt \\&= \int_0^1 \left[ \frac{1}{1+t} - \frac{1}{2+t} \right] \cdot dt \\&= \int_0^1 \frac{1}{1+t} \cdot dt - \int_0^1 \frac{1}{2+t} \cdot dt \\&= [\log|1+t|]_0^1 - [\log|2+t|]_0^1 \\&= [\log(1+1) - \log(1+0)] - [\log(2+1) - \log(2+0)] \\&= \log 2 - \log 3 + \log 2 \quad \dots [\because \log 1 = 0]\end{aligned}$$

$$= \log \left( \frac{2 \times 2}{3} \right)$$

$$= \log \left( \frac{4}{3} \right).$$

Exercise 4.2 | Q 2.08 | Page 172

$$\text{Evaluate : } \int_{-1}^1 \frac{1}{a^2 e^x + b^2 e^{-x}} \cdot dx$$

**SOLUTION**

$$\text{Let } I = \int_{-1}^1 \frac{e^x}{a^2(e^x)^2 + b^2} \cdot dx$$

$$\text{Put } e^x = t$$

$$\therefore e^x \cdot dx = dt$$

$$\text{When } x = 1, t = e$$

$$\text{When } x = -1, t = e^{-1} = \frac{1}{e}$$

$$\begin{aligned}\therefore I &= \int_{\frac{1}{e}}^e \frac{dt}{a^2 t^2 + b^2} \\&= \int_{\frac{1}{e}}^e \frac{dt}{(at)^2 + b^2} \\&= \left[ \frac{1}{a} \cdot \frac{1}{b} \tan^{-1} \left( \frac{at}{b} \right) \right]_{\frac{1}{e}}^e \\&= \frac{1}{ab} \tan^{-1} \left( \frac{ae}{b} \right) - \frac{1}{ab} \tan^{-1} \left( \frac{a}{be} \right) \\&= (1)ab \left[ \tan^{-1} \left( \frac{ae}{b} \right) - \tan^{-1} \left( \frac{a}{be} \right) \right].\end{aligned}$$

Exercise 4.2 | Q 2.09 | Page 172

$$\text{Evaluate : } \int_0^{\pi} \frac{1}{3 + 2 \sin x + \cos x} \cdot dx$$

**SOLUTION**

$$\text{Let } I = \int_0^{\pi} \frac{1}{3 + 2 \sin x + \cos x} \cdot dx$$

$$\text{Put } \tan \frac{x}{2} = t$$

$$\therefore x = 2 \tan^{-1} t$$

$$\therefore dx = \frac{2dt}{1+t^2}$$

and

$$\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$$

$$\text{When } x = 0, t = \tan 0 = 0$$

$$\text{When } x = \pi, t = \tan \frac{\pi}{2} = \infty$$

$$\therefore I = \int_0^{\infty} \frac{1}{3 + 2\left(\frac{2t}{1+t^2}\right) + \left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^{\infty} \frac{1}{2t^2 + 4t + 4} \cdot dt$$

$$= \frac{2}{2} \int_0^{\infty} \frac{1}{t^2 + 2t + 2} \cdot dt$$

$$= \int_0^{\infty} \frac{1}{(t^2 + 2t + 1) + 1} \cdot dt$$

$$= \int_0^{\infty} \frac{1}{(t^2 + 2t + 1 + 1) \cdot dt}$$

$$= \int_0^{\infty} \frac{1}{(t+1)^2 + (1)^2} \cdot dt$$

$$= \frac{1}{1} \left[ \tan^{-1} \left( \frac{t+1}{1} \right) \right]_0^{\infty}$$

$$\begin{aligned}
 &= \left[ \tan^{-1}(t+1) \right]_0^{\infty} \\
 &= \tan^{-1} \infty - \tan^{-1} 1 \\
 &= \frac{\pi}{2} - \frac{\pi}{4} \\
 &= \frac{\pi}{4}.
 \end{aligned}$$

Exercise 4.2 | Q 2.1 | Page 172

Evaluate :  $\int_0^{\frac{\pi}{4}} \sec^4 x \cdot dx$

**SOLUTION**

$$\begin{aligned}
 \text{Let } I &= \int_0^{\frac{\pi}{4}} \sec^4 x \cdot dx \\
 &= \int_0^{\frac{\pi}{4}} \sec^2 x \cdot \sec^2 x \cdot dx \\
 &= \int_0^{\frac{\pi}{4}} (1 + \tan^2 x) \sec^2 x \cdot dx
 \end{aligned}$$

Put  $\tan x = t$

$$\therefore \sec^2 x \cdot dx = dt$$

When  $x = 0$ ,  $t = \tan 0 = 0$

When  $x = \frac{\pi}{4}$ ,  $t = \tan \frac{\pi}{4} = 1$

$$\begin{aligned}
 \therefore I &= \int_0^1 (1 + t^2) \cdot dt \\
 &= \left[ t + \frac{t^3}{3} \right]_0^1 \\
 &= 1 + \frac{1}{3} - 0
 \end{aligned}$$

$$= \frac{4}{3}.$$

Exercise 4.2 | Q 2.11 | Page 172

Evaluate :  $\int_0^1 \sqrt{\frac{1-x}{1+x}} \cdot dx$

**SOLUTION**

Let  $I = \int_0^1 \sqrt{\frac{1-x}{1+x}} \cdot dx$

Put  $x = \cos \theta$

$dx = -\sin \theta d\theta$

When  $x = 0$ ,  $\cos \theta = 0 = \cos \frac{\pi}{2} \therefore \theta = \frac{\pi}{2}$

When  $x = 1$ ,  $\cos \theta = 1 = \cos 0 \therefore \theta = 0$

$$\therefore I = \int_{\frac{\pi}{2}}^0 \sqrt{\frac{-\cos \theta}{1 + \cos \theta}} \cdot (-\sin \theta) d\theta$$

$$= \int_{\frac{\pi}{2}}^0 \sqrt{\frac{2 \sin^2(\frac{\theta}{2})}{2 \cos^2(\frac{\theta}{2})}} \left( -2 \frac{\sin \theta}{2} \cos \frac{\theta}{2} \right) \cdot d\theta$$

$$= \int_{\frac{\pi}{2}}^0 \left( \frac{\sin(\frac{\theta}{2})}{\cos(\frac{\theta}{2})} \right) \left[ -2 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \right] \cdot d\theta$$

$$= \int_{\frac{\pi}{2}}^0 -2 \sin^2\left(\frac{\theta}{2}\right) \cdot d\theta$$

$$= - \int_{\frac{\pi}{2}}^0 (1 - \cos \theta) \cdot d\theta$$

$$\begin{aligned}
&= -[\theta - \sin \theta]_{\frac{\pi}{2}}^0 \\
&= -\left[(0 - \sin 0) - \left(\frac{\pi}{2} - \frac{\sin \pi}{2}\right)\right] \\
&= -\left[0 - \frac{\pi}{2} + 1\right] \\
&= \frac{\pi}{2} - 1.
\end{aligned}$$

Exercise 4.2 | Q 2.12 | Page 172

Evaluate :  $\int_0^{\pi} \sin^3 x (1 + 2 \cos x)(1 + \cos x)^2 \cdot dx$

**SOLUTION**

$$\begin{aligned}
\text{Let } I &= \int_0^{\pi} \sin^3 x (1 + 2 \cos x)(1 + \cos x)^2 \cdot dx \\
&= \int_0^{\pi} \sin^2 x (1 + 2 \cos x)(1 + \cos x)^2 \cdot \sin x \cdot dx \\
&= \int_0^{\pi} (1 - \cos^2 x)(1 + 2 \cos x)(1 + \cos x)^2 \cdot \sin x \cdot dx
\end{aligned}$$

Put  $\cos x = t$

$$\therefore -\sin x \cdot dx = dt.$$

$$\therefore \sin x \cdot dx = -dt$$

$$\text{When } x = 0, t = \cos 0 = 1$$

$$\text{When } x = \pi, t = \cos \pi = -1$$

$$\begin{aligned}
\therefore I &= \int_1^{-1} (1 - t^2)(1 + 2t)(1 + t)^2(-dt) \\
&= - \int_1^{-1} (1 + 2t - t^2 - 2t^3)(1 + 2t + t^2) \cdot dt
\end{aligned}$$



$$\begin{aligned}
&= - \int_1^{-1} (1 + 2t - t^2 - 2t^3 + 2t + 4t^2 - 2t^3 - 4t^4 + t^2 + 2t^3 - t^4 - 2t^5) \cdot dt \\
&= \int_1^{-1} (1 + 4t + 4t^2 - 2t^3 - 5t^4 - 2t^5) \cdot dt \\
&= \int_1^{-1} (1 + 4t + 4t^2 - 2t^3 - 5t^4 - 2t^5) \cdot dt \\
&= - \left[ t + 4 \left( \frac{t^2}{2} \right) + 4 \left( \frac{t^3}{3} \right) - 2 \left( \frac{t^4}{4} \right) - 5 \left( \frac{t^5}{5} \right) - 2 \left( \frac{t^6}{6} \right) \right]_1^{-1} \\
&= - \left[ t + 2t^2 + \frac{4}{3}t^3 - \frac{1}{2}t^4 - t^5 - \frac{1}{3}t^6 \right]_1^{-1} \\
&= - \left[ \left( -1 + 2 - \frac{4}{3} - \frac{1}{2} + 1 - \frac{1}{3} \right) - \left( 1 + 2 + \frac{4}{3} - \frac{1}{2} - 1 - \frac{1}{3} \right) \right] \\
&= - \left[ -1 + 2 - \frac{4}{3} - \frac{1}{2} + 1 - \frac{1}{3} - 1 - 2 - \frac{4}{3} + \frac{1}{2} + 1 + \frac{1}{3} \right] \\
&= - \left[ -\frac{8}{3} \right] \\
&= \frac{8}{3}.
\end{aligned}$$

Exercise 4.2 | Q 2.13 | Page 172

Evaluate :  $\int_0^{\frac{\pi}{2}} \sin 2x \cdot \tan^{-1}(\sin x) \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sin 2x \cdot \tan^{-1}(\sin x) \cdot dx$$

$$= \int_0^{\frac{\pi}{2}} \tan^{-1}(\sin x) \cdot (2 \sin x \cos x) \cdot dx$$

Put  $\sin x = t$

$$\therefore \cos x \cdot dx = dt$$

When  $x = 0$ ,  $t = \sin 0 = 0$ .

$$\text{When } x = \frac{\pi}{2}, t = \sin \frac{\pi}{2} = 1$$

$$\therefore I = \int_0^1 (\tan^{-1} t)(2t) \cdot dt$$

$$= \left[ \tan^{-1} t \int 2t \, dt \right]_0^1 - \int_0^1 \left( \frac{d}{dt} (\tan^{-1} t) \int 2t \, dt \right) \cdot dt$$

$$= \left[ \tan^{-1} t (t^2) \right]_0^1 - \int_0^1 \frac{1}{1+t^2} \cdot t^2 \cdot dt$$

$$= t^2 \tan^{-1} t \Big|_0^1 - \int_0^1 \frac{(1+t^2) - 1}{1+t^2} \cdot dt$$

$$= t^2 \tan^{-1} t \Big|_0^1 - \int_0^1 \frac{(1+t^2) - 1}{1+t^2} \cdot dt$$

$$= [1 \cdot \tan^{-1} 1 - 0] - \int_0^1 \left( 1 - \frac{1}{1+t^2} \right) \cdot dt$$

$$= \frac{\pi}{4} - [t - \tan^{-1} t]_0^1$$



$$\begin{aligned}
 &= \frac{p}{4} - [(1 - \tan^{-1} 1) - 0] \\
 &= \frac{\pi}{4} - 1 + \frac{\pi}{4} \\
 &= \frac{\pi}{2} - 1.
 \end{aligned}$$

Exercise 4.2 | Q 2.14 | Page 172

Evaluate :  $\int_{\frac{1}{\sqrt{2}}}^1 \frac{e^{\cos^{-1} x} \sin^{-1} x}{\sqrt{1-x^2}} \cdot dx$

**SOLUTION**

Let  $I = \int_{\frac{1}{\sqrt{2}}}^1 \frac{e^{\cos^{-1} x} \sin^{-1} x}{\sqrt{1-x^2}} \cdot dx$

Put  $\sin^{-1} x = t$

$$\therefore \frac{1}{\sqrt{1-x^2}} \cdot dx = dt$$

When  $x = 1$ ,  $t = \sin^{-1} 1 = \frac{\pi}{2}$

When  $x = \frac{1}{\sqrt{2}}$ ,  $t = \frac{\sin^{-1} 1}{\sqrt{2}} = \frac{\pi}{4}$

Also,  $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x = \frac{\pi}{2} - t$

$$\therefore I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{\frac{\pi}{2}-t} \cdot t \, dt$$

$$= e^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} t e^{-t} dt$$

$$= e^{\frac{\pi}{2}} \left\{ \left[ t \int e^{-t} dt \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[ \frac{d}{dt}(t) \int e^{-t} dt \right] \cdot dt \right\}$$

$$\begin{aligned}
&= e^{\frac{\pi}{2}} \left\{ \left[ -te^{-t} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1)(-e^{-t}) \cdot dt \right\} \\
&= e^{\frac{\pi}{2}} \left\{ \frac{-\pi}{2} e^{-\frac{\pi}{2}} + \frac{\pi}{4} e^{-\frac{\pi}{4}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{-t} \cdot dt \right\} \\
&= -\frac{\pi}{2} e^o + \frac{\pi}{4} e^{\frac{\pi}{2} - \frac{\pi}{4}} + e^{\frac{\pi}{2}} \left[ -e^{-t} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= -\frac{\pi}{2} + \frac{\pi}{4} e^{\frac{\pi}{4}} + e^{\frac{\pi}{2}} \left[ -e^{-\frac{\pi}{2}} + e^{-\frac{\pi}{4}} \right] \\
&= -\frac{\pi}{2} + e^{\frac{\pi}{4}} \frac{\pi}{4} - e^o + \frac{\pi}{2} - \frac{\pi}{4} \\
&= -\frac{\pi}{2} + e^{\frac{\pi}{4}} \frac{\pi}{4} - 1 + e^{\frac{\pi}{4}} \\
&= e^{\frac{\pi}{4}} \left( \frac{\pi}{4} + 1 \right) - \left( \frac{\pi}{2} + 1 \right).
\end{aligned}$$

Exercise 4.2 | Q 2.15 | Page 172

Evaluate :  $\int_1^3 \frac{\cos(\log x)}{x} \cdot dx$

**SOLUTION**

Let  $I = \int_1^3 \frac{\cos(\log x)}{x} \cdot dx$

$= \int_1^3 \cos(\log x) \cdot \frac{1}{x} \cdot dx$

Put  $\log x = t$

$\therefore \frac{1}{x} \cdot dx = dt$

When  $x = 1$ ,  $t = \log 1 = 0$

When  $x = 3$ ,  $t = \log 3$

$$\begin{aligned}\therefore I &= \int_0^{\log 3} \cos t \cdot dt = [\sin t]_0^{\log 3} \\ &= \sin (\log 3) - \sin 0 \\ &= \sin (\log 3).\end{aligned}$$

Exercise 4.2 | Q 3.01 | Page 172

Evaluate the following:  $\int_0^a \frac{1}{x + \sqrt{a^2 - x^2}} \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_0^a \frac{1}{x + \sqrt{a^2 - x^2}} \cdot dx$$

Put  $x = a \sin \theta$

$$\therefore dx = a \cos \theta d\theta$$

and

$$\sqrt{a^2 - x^2}$$

$$= \sqrt{a^2 - a^2 \sin^2 \theta}$$

$$= \sqrt{a^2 (1 - \sin^2 \theta)}$$

$$= \sqrt{a^2 \cos^2 \theta}$$

$$= a \cos \theta$$

$$\text{When } x = 0, a \sin \theta = 0 \quad \therefore \theta = 0$$

$$\text{When } x = a, a \sin \theta = a \quad \therefore \theta = \frac{\pi}{2}$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{a \cos \theta d\theta}{a \sin \theta + a \cos \theta}$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \cos \theta} \cdot d\theta \quad \dots(1)$$

We use the property,  $\int_0^a f(a-x) \cdot dx$ .

Hence in I, we change  $\theta$  by  $\left(\frac{\pi}{2}\right) - \theta$ .

$$\begin{aligned}\therefore I &= \int_0^{\frac{\pi}{2}} \frac{\cos\left[\left(\frac{\pi}{2}\right) - \theta\right]}{\sin\left[\left(\frac{\pi}{2}\right) - \theta\right] + \cos\left[\left(\frac{\pi}{2}\right) - \theta\right]} \cdot d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\cos \theta + \sin \theta} \cdot d\theta \quad \dots(2)\end{aligned}$$

Adding (1) and (2), we get

$$\begin{aligned}2I &= \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + \cos \theta} \cdot d\theta + \int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\cos \theta + \sin \theta} \cdot d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos \theta + \sin \theta}{\cos \theta + \sin \theta} \cdot d\theta \\ &= \int_0^{\frac{\pi}{2}} 1 \cdot d\theta = [\theta]_0^{\frac{\pi}{2}} \\ &= \left(\frac{\pi}{2}\right) - 0 \\ &= \frac{\pi}{2} \\ \therefore I &= \frac{\pi}{4}.\end{aligned}$$

Exercise 4.2 | Q 3.02 | Page 172

Evaluate the following :  $\int_0^{\frac{\pi}{2}} \log(\tan x) \cdot dx$

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**SOLUTION**

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \log(\tan x) \cdot dx$$

$$\text{We use the property, } \int_0^a f(x) \cdot dx = \int_0^a f(a-x) \cdot dx.$$

$$\text{Here, } a = \frac{\pi}{2}.$$

Hence changing  $x$  by  $\frac{\pi}{2} - x$ , we get

$$I = \int_0^{\frac{\pi}{2}} \log\left[\tan\left(\frac{\pi}{2} - x\right)\right] \cdot dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\cot x) \cdot dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{1}{\tan x}\right) \cdot dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\tan x)^{-1} \cdot dx$$

$$= \int_0^{\frac{\pi}{2}} -\log(\tan x) \cdot dx$$

$$= - \int_0^{\frac{\pi}{2}} \log(\tan x) \cdot dx$$

$$= -I$$

$$\therefore 2I = 0$$

$$\therefore I = 0.$$

Exercise 4.2 | Q 3.03 | Page 172

$$\text{Evaluate the following : } \int_0^1 \log\left(\frac{1}{x} - 1\right) \cdot dx$$



**SOLUTION**

$$\begin{aligned}\text{Let } I &= \int_0^1 \log\left(\frac{1}{x} - 1\right) \cdot dx \\&= \int_0^1 \log\left(\frac{1-x}{x}\right) \cdot dx \\&= \int_0^1 [\log(1-x) - \log x] \cdot dx \quad \dots(1)\end{aligned}$$

We use the property  $\int_0^a f(x) \cdot dx = \int_0^a f(a-x) \cdot dx$

Here,  $a = 1$

Hence in  $I$ , changing  $x$  to  $1-x$ , we get

$$\begin{aligned}I &= \int_0^1 [\log|1 - (1-x)| - \log(1-x)] \cdot dx \\&= \int_0^1 [\log x - \log(1-x)] \cdot dx \\&= - \int_0^1 [\log(1-x) - \log x] \cdot dx \\&= -1 \quad \dots[\text{By (1)}] \\ \therefore 2I &= 0 \\ \therefore I &= 0.\end{aligned}$$

Exercise 4.2 | Q 3.04 | Page 172

Evaluate :  $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} \cdot dx$





**SOLUTION**

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} \cdot dx$$

$$\text{We use the property, } \int_0^a f(x) \cdot dx = \int_0^a f(a-x) \cdot dx.$$

$$\text{Here } a = \frac{\pi}{2}.$$

$$\text{Hence In I, we change } x \text{ by } \frac{\pi}{2} - x.$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \cos x \sin x} \cdot dx$$

$$= - \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} \cdot dx$$

$$= -I$$

$$\therefore 2I = 0$$

$$\therefore I = 0.$$

**Exercise 4.2 | Q 3.05 | Page 172**

$$\text{Evaluate the following : } \int_0^3 x^2(3-x)^{\frac{5}{2}} \cdot dx$$

**SOLUTION**

$$\text{Let } I = \int_0^3 x^2(3-x)^{\frac{5}{2}} \cdot dx$$

$$\text{We use the property } \int_0^a f(x) \cdot dx = \int_0^a f(a-x) \cdot dx$$



Here,  $a = 3$

Hence in I, changing  $x$  to  $3 - x$ , we get

$$\begin{aligned} I &= \int_0^3 (3-x)^2 [3 - (3-x)]^{\frac{5}{2}} \cdot dx \\ &= \int_0^3 (9 - 6x + x^2) x^{\frac{5}{2}} \cdot dx \\ &= \int_0^3 \left[ 9x^{\frac{5}{2}} - 6x^{\frac{7}{2}} + x^{\frac{9}{2}} \right] \cdot dx \\ &= 9 \int_0^3 x^{\frac{5}{2}} \cdot dx - 6 \int_0^3 x^{\frac{7}{2}} \cdot dx + \int_0^3 x^{\frac{9}{2}} \cdot dx \\ &= 9 \left[ \frac{x^{\frac{7}{2}}}{\frac{7}{2}} \right]_0^3 - 6 \left[ \frac{x^{\frac{9}{2}}}{\frac{9}{2}} \right]_0^3 + 9 \left[ \frac{x^{\frac{11}{2}}}{\frac{11}{2}} \right]_0^3 \\ &= 9 \left[ \frac{2 \cdot 3^{\frac{7}{2}}}{7} - 0 \right] - 6 \left[ \frac{2 \cdot 3^{\frac{9}{2}}}{9} - 0 \right] + \left[ \frac{2}{11} \cdot 3^{\frac{11}{2}} - 0 \right] \\ &= \frac{18}{7} 3^{\frac{7}{2}} - \frac{2 \cdot 6}{9} \cdot 3^{\frac{7}{2}} \cdot 3 + \frac{2}{11} \cdot 3^{\frac{7}{2}} \cdot 3^2 \\ &= 2(3)^{\frac{7}{2}} \left[ \frac{9}{7} - 2 + \frac{9}{11} \right] \\ &= 2(3)^{\frac{7}{2}} \left[ \frac{99 - 154 + 63}{77} \right] \\ &= 2(3)^{\frac{7}{2}} \times \frac{8}{77} \\ &= \frac{16}{77} (3)^{\frac{7}{2}}. \end{aligned}$$

Exercise 4.2 | Q 3.06 | Page 172

Evaluate the following :  $\int_{-3}^3 \frac{x^3}{9-x^2} \cdot dx$

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**SOLUTION**

$$\text{Let } I = \int_{-3}^3 \frac{x^3}{9 - x^2} \cdot dx$$

$$\text{Let } f(x) = \frac{x^3}{9 - x^2}$$

$$\therefore f(-x) = \frac{(-x)^3}{9 - (-x)^2}$$

$$= \frac{-x^3}{9 - x^2}$$

$$= -f(x)$$

$\therefore f$  is an odd function.

$$\therefore \int_{-3}^3 f(x) \cdot dx = 0, \text{ i.e. } \int_{-3}^3 \frac{x^3}{9 - x^2} \cdot dx = 0.$$

**Exercise 4.2 | Q 3.07 | Page 172**

Evaluate the following :  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{2 + \sin x}{2 - \sin x}\right) \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{2 - \sin x}{2 + \sin x}\right) \cdot dx$$

$$\text{Let } f(x) = \log\left(\frac{2 - \sin x}{2 + \sin x}\right)$$

$$\therefore f(-x) = \log\left[\frac{2 - \sin(-x)}{2 + \sin(-x)}\right]$$

$$= \log\left(\frac{2 + \sin x}{2 - \sin x}\right)$$

$$= -\log\left(\frac{2 - \sin x}{2 + \sin x}\right)$$

$$= -f(x)$$

$\therefore f$  is an odd function.

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x) \cdot dx = 0$$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{2 - \sin x}{2 + \sin x}\right) \cdot dx = 0.$$

Exercise 4.2 | Q 3.08 | Page 172

Evaluate the following :  $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} \cdot dx$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[ \frac{x}{2 - \cos 2x} + \frac{\frac{\pi}{4}}{2 - \cos 2x} \right]$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x}{2 - \cos 2x} \cdot dx + \frac{\pi}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2 - \cos 2x} \cdot dx$$

$$= I_1 + \frac{\pi}{4} I_2 \quad \dots(1)$$

$$\text{Let } f(x) = \frac{x}{2 - \cos 2x}$$

$$\therefore f(-x) = \frac{-x}{2 - \cos[2(-x)]}$$

$$= \frac{-x}{2 - \cos 2x}$$

$$= -f(x)$$

$\therefore f$  is an odd function

$$\therefore \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(x) \cdot dx = 0$$

$$\text{i.e. } \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x}{2 - \cos 2x} \cdot dx = 0, \text{ i.e. } I_1 = 0 \quad \dots(2)$$

In  $I_2$ , put  $\tan x = t$

$$\therefore x = \tan^{-1}t$$

$$\therefore dx = \frac{1}{1+t^2} \cdot dt$$

and

$$\cos 2x = \frac{1-t^2}{1+t^2}$$

$$\text{When } x = -\frac{\pi}{4}, t = \tan\left(-\frac{\pi}{4}\right) = -1$$

$$\text{When } x = \frac{\pi}{4}, t = \frac{\tan \pi}{4} = 1.$$

$$\therefore I_2 = \int_{-1}^1 \frac{1}{2 - \left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{1}{1+t^2} \cdot dt$$

$$= \int_{-1}^1 \frac{1}{2(1+t^2) - (1-t^2)} \cdot dt$$

$$\begin{aligned}
&= \int_{-1}^1 \frac{1}{3t^2 + 1} \cdot dt \\
&= \int_{-1}^1 \frac{1}{(\sqrt{3}t)^2 + 1} \\
&= \left[ \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{\sqrt{3}t}{1} \right) \right]_{-1}^1 \\
&= \frac{1}{\sqrt{3}} \left[ \tan^{-1} \sqrt{3} - \tan^{-1} (-\sqrt{3}) \right] \\
&= \frac{1}{\sqrt{3}} \left[ \tan^{-1} \sqrt{3} + \tan^{-1} \sqrt{3} \right] \\
&= \frac{1}{\sqrt{3}} \left[ \frac{\pi}{3} + \frac{\pi}{3} \right] \\
&= \frac{2\pi}{3\sqrt{3}} \quad \dots(3)
\end{aligned}$$

From (1), (2) and (3), we get

$$\begin{aligned}
I &= 0 + \frac{\pi}{4} \left[ \frac{2\pi}{3\sqrt{3}} \right] \\
&= \frac{\pi^2}{6\sqrt{3}}.
\end{aligned}$$

Exercise 4.2 | Q 3.09 | Page 172

Evaluate the following :  $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^3 \sin^4 x \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^3 \sin^4 x \cdot dx$$

$$\text{Let } f(x) = x^3 \sin^4 x$$

$$\therefore f(-x) = (-x)^3 \sin^4(-x)$$

$$= -x^3 \sin^4 x$$

$$= -f(x)$$

$\therefore f$  is an odd function.

$$\therefore \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(x) \cdot dx = 0, \text{ i.e. } \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^3 \sin^4 x \cdot dx = 0.$$

**Exercise 4.2 | Q 3.1 | Page 172**

Evaluate the following :  $\int_0^1 \frac{\log(x+1)}{x^2+1} \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_0^1 \frac{\log(x+1)}{x^2+1} \cdot dx$$

Put  $x = \tan \theta$ .

$$\therefore dx = \sec^2 \theta \cdot d\theta$$

and

$$x^2 + 1 = \tan^2 \theta + 1 = \sec^2 \theta$$

$$\text{When } x = 0, \tan \theta = 0 \quad \therefore \theta = 0$$

$$\text{When } x = 1, \tan \theta = 1 \quad \therefore \theta = \frac{\pi}{4}$$

$$\begin{aligned}\therefore I &= \int_0^{\frac{\pi}{4}} \frac{\log(\tan \theta + 1)}{\sec^2 \theta} \cdot \sec 2\theta \cdot d\theta \\ &= \int_0^{\frac{\pi}{4}} \log(1 + \tan \theta) \cdot d\theta \quad \dots(1)\end{aligned}$$

We use the property,  $\int_0^a f(x) \cdot dx = \int_0^a f(a - x) \cdot dx$ .

Here,  $a = \pi/4$ .

Hence changing  $\theta$  by  $\frac{\pi}{4} - \theta$ , we have,

$$\begin{aligned}I &= \int_0^{\frac{\pi}{4}} \log \left[ 1 + \tan \left( \frac{\pi}{4} - \theta \right) \right] \cdot d\theta \\ &= \int_0^{\frac{\pi}{4}} \log \left( 1 + \frac{1 - \tan \theta}{1 + \tan \theta} \right) \cdot d\theta \\ &= \int_0^{\frac{\pi}{4}} \log \left( \frac{1 + \tan \theta + 1 - \tan \theta}{1 + \tan \theta} \right) \cdot d\theta \\ &= \int_0^{\frac{\pi}{4}} \log \left( \frac{2}{1 + \tan \theta} \right) \cdot d\theta \\ &= \int_0^{\frac{\pi}{4}} [\log 2 - \log(1 + \tan \theta)] \cdot d\theta \\ &= \log 2 \int_0^{\frac{\pi}{4}} 1 \cdot d\theta - \int_0^{\frac{\pi}{4}} \log(1 + \tan \theta) \cdot d\theta \\ &= (\log 2) [\theta]_0^{\frac{\pi}{4}} - I \\ &= \frac{\pi}{4} \log 2 - I \\ \therefore 2I &= \frac{\pi}{4} \log 2 \\ \therefore I &= \frac{\pi}{8} \log 2.\end{aligned}$$



Evaluate the following :  $\int_{-1}^1 \frac{x^3 + 2}{\sqrt{x^2 + 4}} \cdot dx$

**SOLUTION**

$$\begin{aligned}
 \text{Let } I &= \int_{-1}^1 \frac{x^3 + 2}{\sqrt{x^2 + 4}} \cdot dx \\
 &= \int_{-1}^1 \left[ \frac{x^3}{\sqrt{x^2 + 4}} + \frac{2}{\sqrt{x^2 + 4}} \right] \cdot dx \\
 &= \int_{-1}^1 \frac{x^3}{\sqrt{x^2 + 4}} \cdot dx + 2 \int_{-1}^1 \frac{1}{\sqrt{x^2 + 4}} \cdot dx \\
 &= I_1 + 2I_2 \quad \dots(1)
 \end{aligned}$$

$$\begin{aligned}
 \text{Let } f(x) &= \frac{x^3}{\sqrt{x^2 + 4}} \\
 \therefore f(-x) &= \frac{(-x)^3}{\sqrt{(-x)^2 + 4}}
 \end{aligned}$$

$$= \frac{x^3}{\sqrt{x^2 + 4}}$$

$$= -f(x)$$

$\therefore f$  is an odd function.

$$\therefore \int_{-1}^1 \cdot dx = 0, \text{ i.e.}$$

$$I_1 = \int_{-1}^1 \frac{x^3}{\sqrt{x^2 + 4}} \cdot dx = 0 \quad \dots(2)$$

$$\because (-x)^2 = x^2$$

$$\therefore \frac{1}{\sqrt{x^2 + 4}} \text{ is an even function.}$$

$$\therefore \int_{-1}^1 f(x) \cdot dx = 2 \int_0^1 f(x) \cdot dx$$

$$\begin{aligned} \therefore I_2 &= 2 \int_0^1 \frac{1}{\sqrt{x^2 + 4}} \cdot dx \\ &= 2 \left[ \log \left( x + \sqrt{x^2 + 4} \right) \right]_0^1 \\ &= 2 \left[ \log \left( 1 + \sqrt{1 + 4} \right) - \log \left( 0 + \sqrt{0 + 4} \right) \right] \\ &= 2 \left[ \log \left( \sqrt{5} + 1 \right) - \log 2 \right] \\ &= 2 \log \left( \frac{\sqrt{5} + 1}{2} \right) \quad \dots(3) \end{aligned}$$

From (1), (2) and (3), we get

$$\begin{aligned} I &= 0 + 2 \left[ 2 \log \left( \frac{\sqrt{5} + 1}{2} \right) \right] \\ &= 4 \log \left( \frac{\sqrt{5} + 1}{2} \right). \end{aligned}$$

**Exercise 4.2 | Q 3.12 | Page 172**

Evaluate the following :  $\int_{-a}^a \frac{x + x^3}{16 - x^2} \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_{-a}^a \frac{x + x^3}{16 - x^2} \cdot dx$$

$$\text{Let } f(x) = \frac{x + x^3}{16 - x^2}$$

$$\therefore f(-x) = \frac{(-x) + (-x)^3}{16 - (-x)^2}$$

$$= \frac{-(x + x^3)}{16 - x^2}$$

$$= -f(x)$$

$\therefore f$  is an odd function.

$$\therefore \int_{-a}^a f(x) \cdot dx = 0, \text{ i.e. } \int_a^a \frac{x + x^3}{16 - x^2} \cdot dx = 0.$$

**Exercise 4.2 | Q 3.13 | Page 172**

Evaluate the following :  $\int_0^1 t^2 \sqrt{1-t} \cdot dt$

**SOLUTION**

We use the property,

$$\int_0^a f(t) \cdot dt = \int_0^a f(a-t) \cdot dt$$

$$\therefore \int_0^1 t^2 \sqrt{t}(1-t) \cdot dt = \int_0^1 (1-t)^2 \sqrt{1-1+t} \cdot dt$$

$$= \int_0^1 (1-2t+t^2) \sqrt{t} \cdot dt$$

$$\begin{aligned}
&= \int_0^1 \left( t^{\frac{1}{2}} - 2t^{\frac{3}{2}} + t^{\frac{5}{2}} \right) \cdot dt \\
&= \left[ \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 2 \cdot \frac{t^{\frac{5}{2}}}{\frac{5}{2}} + \frac{t^{\frac{7}{2}}}{\frac{7}{2}} \right]_0^1 \\
&= \frac{2}{3} (1)^{\frac{3}{2}} - \frac{4}{5} (1)^{\frac{5}{2}} + \frac{2}{7} (1)^{\frac{7}{2}} - 0 \\
&= \frac{2}{3} - \frac{4}{5} + \frac{2}{7} - 0 \\
&= \frac{70 - 84 + 30}{105} \\
&= \frac{16}{105}.
\end{aligned}$$

Exercise 4.2 | Q 3.14 | Page 172

Evaluate the following :  $\int_0^{\pi} x \sin x \cos^2 x \cdot dx$

**SOLUTION**

$$\begin{aligned}
\text{Let } I &= \int_0^{\pi} x \sin x \cos^2 x \cdot dx \\
&= \frac{1}{2} \int_0^{\pi} x (2 \sin x \cos x) \cos x \cdot dx \\
&= \frac{1}{2} \int_0^{\pi} x (\sin 2x \cos x) \cdot dx \\
&= \frac{1}{4} \int_0^{\pi} x (2 \sin 2x \cos x) \cdot dx \\
&= \frac{1}{4} \int_0^{\pi} [\sin(2x + x) + \sin(2x - x)] \cdot dx \\
&= \frac{1}{4} \left[ \int_0^{\pi} x \sin 3x \cdot dx + \int_0^{\pi} x \sin x \cdot dx \right]
\end{aligned}$$

$$= \frac{1}{4} [I_1 + I_2] \quad \dots(1)$$

$$\begin{aligned} I_1 &= \int_0^{\pi} x \sin 3x \cdot dx \\ &= \left[ x \int \sin 3x \cdot dx \right]_0^{\pi} - \int \left[ \left\{ \frac{d}{dx}(x) \int \sin 3x \cdot dx \right\} \right] \cdot dx \\ &= \left[ x \left( \frac{-\cos 3x}{3} \right) \right]_0^{\pi} - \int_0^{\pi} 1 \left( \frac{-\cos 3x}{3} \right) \cdot dx \\ &= \left[ -\frac{\pi \cos 3\pi}{3} + 0 \right] + \frac{1}{3} \int_0^{\pi} \cos 3x \cdot dx \\ &= -\frac{\pi}{3}(-1) + \frac{1}{3} \left[ \frac{\sin 3x}{3} \right]_0^{\pi} \\ &= \frac{\pi}{3} + \frac{1}{3} [0 - 0] \\ &= \frac{\pi}{3} \quad \dots(2) \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^{\pi} x \sin x \cdot dx \\ &= \left[ x \int \sin x \cdot dx \right]_0^{\pi} - \int_0^{\pi} \left[ \left\{ \frac{d}{dx}(x) \int \sin x \cdot dx \right\} \right] \cdot dx \\ &= [x(-\cos x)]_0^{\pi} - \int_0^{\pi} 1 \cdot (-\cos x) \cdot dx \\ &= [-\pi \cos \pi + 0] + \int_0^{\pi} \cos x \cdot dx \end{aligned}$$

$$\begin{aligned}
 &= -\pi(-1) + [\sin x]_0^\pi \\
 &= \pi + [\sin \pi - \sin 0] \\
 &= \pi + (0 - 0) \\
 &= \pi \quad \dots(3)
 \end{aligned}$$

From (1), (2) and (3), we get

$$\begin{aligned}
 I &= \frac{1}{4} \left[ \frac{\pi}{3} + \pi \right] \\
 &= \frac{1}{4} \left( \frac{4\pi}{3} \right) \\
 &= \frac{\pi}{3}.
 \end{aligned}$$

Exercise 4.2 | Q 3.15 | Page 172

Evaluate the following :  $\int_0^1 \frac{\log x}{\sqrt{1-x^2}} \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_0^1 \frac{\log x}{\sqrt{1-x^2}} \cdot dx$$

Put  $x = \sin \theta$

$$\therefore dx = \cos \theta \, d\theta$$

and

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta$$

When  $x = 0$ ,  $\sin \theta = 0 \therefore \theta = 0$

When  $x = 1$ ,  $\sin \theta = 1 \therefore \theta = \frac{\pi}{2}$

$$\therefore I = \int_0^{\frac{\pi}{2}} \log \sin \theta \cdot d\theta$$

Using the property,  $\int_0^{2a} f(x) \cdot dx = \int_0^a [f(x) + f(2a - x)] \cdot dx$ , we get

$$I = \int_0^{\frac{\pi}{4}} \left[ \log \sin \theta + \log \sin \left( \frac{\pi}{2} - \theta \right) \right] \cdot d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\log \sin \theta + \log \cos \theta) \cdot d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log \sin \theta \cos \theta \cdot d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log \left( \frac{2 \sin \theta \cos \theta}{2} \right) \cdot d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\log \sin 2\theta - \log 2) \cdot d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log \sin 2\theta \cdot d\theta - \int_0^{\frac{\pi}{4}} \log 2 \cdot d\theta$$

$$= I_1 - I_2 \quad \dots(\text{Say})$$

$$I_2 = \int_0^{\frac{\pi}{4}} \log 2 \cdot d\theta$$

$$= \log 2 \int_0^{\frac{\pi}{4}} 1 \cdot d\theta$$

$$= \log 2 [\theta]_0^{\frac{\pi}{4}}$$

$$= (\log 2) \left[ \frac{\pi}{4} - 0 \right]$$

$$= \frac{\pi}{4} \log 2$$

$$I_1 = \int_0^{\frac{\pi}{4}} \log \sin 2\theta \cdot d\theta$$

Put  $2\theta = t$ .

$$\text{Then } d\theta = \frac{dt}{2}$$

When  $\theta = 0$ ,  $t = 0$

$$\text{When } \theta = \frac{\pi}{4}, t = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$$

$$\therefore I_1 = \int_0^{\frac{\pi}{2}} \log \sin t \times \frac{dt}{2}$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \log \sin \theta \cdot d\theta$$

$$= \frac{1}{2} I \quad \dots \left[ \because \int_a^b f(x) \cdot dx = \int_a^b f(t) \cdot dt \right]$$

$$\therefore I = \frac{1}{2} I - \frac{\pi}{4} \log 2$$

$$\therefore I = \frac{1}{2} I - \frac{\pi}{4} \log 2$$

$$\therefore \frac{1}{2} I = -\frac{\pi}{4} \log 2$$

$$\therefore I = -\frac{\pi}{2} \log 2$$

$$= \frac{\pi}{2} \log \left( \frac{1}{2} \right).$$



## MISCELLANEOUS EXERCISE 4 [PAGES 175 - 177]

Miscellaneous Exercise 4 | Q 1.01 | Page 175

Choose the correct option from the given alternatives :

$$\int_2^3 \frac{dx}{x(x^3 - 1)} =$$

$\frac{1}{3} \log\left(\frac{208}{189}\right)$

$\frac{1}{3} \log\left(\frac{189}{208}\right)$

$\log\left(\frac{208}{189}\right)$

$\log\left(\frac{189}{208}\right)$

**SOLUTION**

$$\frac{1}{3} \log\left(\frac{208}{189}\right)$$

Miscellaneous Exercise 4 | Q 1.02 | Page 175

Choose the correct option from the given alternatives :

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x \cdot dx}{(1 + \cos x)^2} =$$

$\frac{4 - \pi}{2}$

$\frac{\pi - 4}{2}$

$4 - \frac{\pi}{2}$

$\frac{4 + \pi}{2}$



**SOLUTION**

$$\frac{4 - \pi}{2}$$

Miscellaneous Exercise 4 | Q 1.03 | Page 175

Choose the correct option from the given alternatives :

$$\int_0^{\log 5} \frac{e^x \sqrt{e^x - 1}}{e^x + 3} \cdot dx =$$

$3 + 2\pi$   
 $2 + \pi$   
 **$4 - \pi$**   
 $4 + \pi$

**SOLUTION**

$$4 - \pi$$

Miscellaneous Exercise 4 | Q 1.04 | Page 175

Choose the correct option from the given alternatives :

$$\int_0^{\frac{\pi}{2}} \sin^6 x \cos^2 x \cdot dx =$$

$\frac{7\pi}{256}$   
 $\frac{3\pi}{256}$   
 **$\frac{5\pi}{256}$**   
 $\frac{-5\pi}{256}$

**SOLUTION**

$$\frac{5\pi}{256}$$

Choose the correct option from the given alternatives :

If  $\frac{dx}{\sqrt{1+x} - \sqrt{x}} = \frac{k}{3}$ , then k is equal to

$\sqrt{2}(2\sqrt{2} - 2)$   
 $\frac{\sqrt{2}}{3}(2 - 2\sqrt{2})$   
 $\frac{2\sqrt{2} - 2}{3}$   
 $4\sqrt{2}$

**SOLUTION**

$$4\sqrt{2}$$

Choose the correct option from the given alternatives :

$\int_1^2 \frac{1}{x^2} e^{\frac{1}{x}} \cdot dx =$

$\sqrt{e} + 1$   
 $\sqrt{e} - 1$   
 $\sqrt{e}(\sqrt{e} - 1)$   
 $\frac{\sqrt{e} - 1}{e}$

**SOLUTION**

$$\sqrt{e}(\sqrt{e} - 1)$$

Choose the correct option from the given alternatives :

If  $\left[ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right] \cdot dx = a + \frac{b}{\log 2}$ , then

**a = e, b = -2**

a = e, b = 2

a = -e, b = 2

a = -e, b = -2

**SOLUTION**

a = e, b = -2

Choose the correct option from the given alternatives :

Let  $I_1 = \int_e^{e^2} \frac{dx}{\log x}$  and  $I_2 = \int_1^2 \frac{e^x}{x} \cdot dx$ , then

$I_1 = \frac{1}{3} I_2$

$I_1 + I_2 = 0$

$I_1 = 2I_2$

**$I_1 = I_2$**

**SOLUTION**

$I_1 = I_2$

Choose the correct option from the given alternatives :

$$\int_0^9 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{9-x}} \cdot dx =$$

9

 $\frac{9}{2}$ 

2

0

1

**SOLUTION**

9/2

Choose the correct option from the given alternatives :

The value of  $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log \left( \frac{2 + \sin \theta}{2 - \sin \theta} \right) \cdot d\theta$  is

0

1

2

 $\pi$ 

**SOLUTION**

0

Evaluate the following :  $\int_0^{\frac{\pi}{2}} \frac{\cos x}{3 \cos x + \sin x} \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{3 \cos x + \sin x} \cdot dx$$

$$\text{Put Numerator} = A(\text{Denominator}) + B \left[ \frac{d}{dx}(\text{Denominator}) \right]$$

$$\therefore \cos x = A(3 \cos x + \sin x) + B \left[ \frac{d}{dx}(3 \cos x + \sin x) \right]$$

$$= A(3 \cos x + \sin x) + B(-3 \sin x + \cos x)$$

$$\therefore \cos x + 0 \cdot \sin x = (3A + B)\cos x + (A - 3B)\sin x$$

Comparing the coefficient of  $\sin x$  and  $\cos x$  on both the sides, we get

$$3A + B = 1 \quad \dots(1)$$

$$A - 3B = 0 \quad \dots(2)$$

Multiplying equation (1) by 3, we get

$$9A + 3B = 3 \quad \dots(3)$$

Adding (2) and (3), we get

$$10A = 3$$

$$\therefore A = \frac{3}{10}$$

$$\therefore \text{from (1), } 3\left(\frac{3}{10}\right) + B = 1$$

$$\therefore B = 1 - \frac{9}{10} = \frac{1}{10}$$

$$\therefore \cos x = \frac{3}{10}(3 \cos x + \sin x) + \frac{1}{10}(-3 \sin x + \cos x)$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \left[ \frac{\frac{3}{10}(3 \cos x + \sin x) + \frac{1}{10}(-3 \sin x + \cos x)}{3 \cos x + \sin x} \right] \cdot dx$$

$$= \int_0^{\frac{\pi}{2}} \left[ \frac{3}{10} + \frac{\frac{1}{10}(-3 \sin x + \cos x)}{3 \cos x + \sin x} \right] \cdot dx$$

$$\begin{aligned}
&= \frac{3}{10} \int_0^{\frac{\pi}{2}} 1 \cdot dx + \frac{1}{10} \int_0^{\frac{\pi}{2}} \frac{-3 \sin x + \cos x}{3 \cos x + \sin x} \cdot dx \\
&= \frac{3}{10} \int_0^{\frac{\pi}{2}} + \frac{1}{10} [\log |3 \cos x + \sin x|]_0^{\frac{\pi}{2}} \quad \dots \left[ \because \int \frac{f'(x)}{f(x)} \cdot dx = \log |f(x)| + c \right] \\
&= \frac{3}{10} \left[ \frac{\pi}{2} - 0 \right] + \frac{1}{10} \left[ \log \left| 3 \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \right| - \log |3 \cos 0 + \sin 0| \right] \\
&= \frac{3\pi}{20} + \frac{1}{10} [\log |3 \times 0 + 1| - \log |3 \times 1 + 0|] \\
&= \frac{3\pi}{20} + \frac{1}{10} [\log 1 - \log 3] \\
&= \frac{3\pi}{20} - \frac{1}{10} \log 3. \quad \dots [\because \log 1 = 0]
\end{aligned}$$

Miscellaneous Exercise 4 | Q 2.02 | Page 176

Evaluate the following :  $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos \theta}{\left[ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right]^3} \cdot d\theta$

**SOLUTION**

$$\begin{aligned}
\text{Let } I &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos \theta}{\left[ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right]^3} \cdot d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}}{\left[ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right]^3} \cdot d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\left( \cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right) \left( \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right)}{\left[ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right]^3} \cdot d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}}{\left[ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right]^2} \cdot d\theta
\end{aligned}$$



$$\text{Put } \cos \frac{\theta}{2} - \sin \frac{\theta}{2} = t$$

$$\therefore \left( -\frac{1}{2} \sin \frac{\theta}{2} + \frac{1}{2} \cos \frac{\theta}{2} \right) \cdot d\theta = dt$$

$$\therefore \left( \cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right) \cdot d\theta = 2 \cdot dt$$

$$\text{When } \theta = \frac{\pi}{4}, t = \cos \frac{\pi}{8} + \sin \frac{\pi}{8}$$

$$\text{When } \theta = \frac{\pi}{2}, t = \cos \frac{\pi}{4} + \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

$$\therefore I = \int_{\cos \frac{\pi}{8} + \sin \frac{\pi}{8}}^{\sqrt{2}} \frac{1}{t^2} \cdot 2dt$$

$$= 2 \int_{\cos \frac{\pi}{8} + \sin \frac{\pi}{8}}^{\sqrt{2}} t^{-2} \cdot dt$$

$$= 2 \left[ \frac{t^{-1}}{-1} \right]_{\cos \frac{\pi}{8} + \sin \frac{\pi}{8}}^{\sqrt{2}}$$

$$= \left[ \frac{-2}{t} \right]_{\cos \frac{\pi}{8} + \sin \frac{\pi}{8}}^{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}} + \frac{2}{\cos \frac{\pi}{8} + \sin \frac{\pi}{8}}$$

$$= \frac{2}{\cos \frac{\pi}{8} + \sin \frac{\pi}{8}} - \sqrt{2}.$$

Miscellaneous Exercise 4 | Q 2.03 | Page 176

Evaluate the following :  $\int_0^1 \frac{1}{1 + \sqrt{x}} \cdot dx$



**SOLUTION**

$$\text{Let } I = \int_0^1 \frac{1}{1 + \sqrt{x}} \cdot dx$$

$$\text{Put } \sqrt{x} = t$$

$$\therefore x = t^2 \text{ and } dx = 2t \cdot dt$$

$$\text{When } x = 0, t = 0$$

$$\text{When } x = 1, t = 1$$

$$\therefore I = \int_0^1 \frac{1}{1 + t} 2t \cdot dt$$

$$= 2 \int_0^1 \frac{t}{1 + t} \cdot dt$$

$$= 2 \int_0^1 \frac{(1 + t) - 1}{1 + t} \cdot dt$$

$$= 2 \int_0^1 \left( 1 - \frac{1}{1 + t} \right) \cdot dt$$

$$= 2[t - \log|1 + t|]_0^1$$

$$= 2[1 - \log 2 - 0 + \log 1]$$

$$= 2(1 - \log 2) \quad \dots[\because \log 1 = 0]$$

$$= 2 - 2\log 2$$

$$= 2 - \log 4.$$

Miscellaneous Exercise 4 | Q 2.04 | Page 176

Evaluate the following :  $\int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{1 + \cos 2x} \cdot dx$

**SOLUTION**

$$\begin{aligned}\text{Let } I &= \int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{1 + \cos 2x} \cdot dx \\&= \int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{2 \cos^2 x} \cdot dx \\&= \frac{1}{2} \int_0^{\frac{\pi}{4}} \tan^3 x \cdot \sec^2 x \cdot dx\end{aligned}$$

Put  $\tan x = t$

$$\therefore \sec^2 x \cdot dx = dt$$

When  $x = 0$ ,  $t = \tan 0 = 0$

$$\text{When } x = \frac{\pi}{4}, t = \tan \frac{\pi}{4} = 1$$

$$\begin{aligned}\therefore I &= \frac{1}{2} \int_0^1 t^3 \cdot dt \\&= \frac{1}{2} \cdot \left[ \frac{t^4}{4} \right]_0^1 \\&= \frac{1}{8} [t^4]_0^1 \\&= \frac{1}{8} [1 - 0] \\&= \frac{1}{8}.\end{aligned}$$

Miscellaneous Exercise 4 | Q 2.05 | Page 176

Evaluate the following :  $\int_0^1 t^5 \sqrt{1 - t^2} \cdot dt$

**SOLUTION**

$$\text{Let } I = \int_0^1 t^5 \sqrt{1-t^2} \cdot dt$$

$$\text{Put } t = \sin \theta$$

$$\therefore dt = \cos \theta d\theta$$

$$\text{When } t = 1, \theta = \sin^{-1} 1 = \frac{\pi}{2}$$

$$\text{When } t = 0, \theta = \sin^{-1} 0 = 0$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \sin^5 \theta \sqrt{1-\sin^2 \theta} \cos \theta \cdot d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \sin^5 \theta \cdot \cos \theta \cdot \cos \theta \cdot d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^5 \theta (1 - \sin^2 \theta) \cdot d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin^5 \theta - \sin^7 \theta) \cdot d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^5 \theta \cdot d\theta - \int_0^{\frac{\pi}{2}} \sin^7 \theta d\theta.$$

Using Reduction formula, we get

$$I = \frac{4}{5} \cdot \frac{2}{3} - \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

$$= \frac{8}{15} \left[ 1 - \frac{6}{7} \right]$$

$$= \frac{8}{15} \times \frac{1}{7}$$

$$= \frac{8}{105}.$$

Evaluate the following :  $\int_0^1 (\cos^{-1} x^2) \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_0^1 (\cos^{-1} x^2) \cdot dx$$

$$\text{Put } \cos^{-1} x = t$$

$$\therefore x = \cos t$$

$$\therefore dx = -\sin t \cdot dt$$

$$\text{When } x = 0, t = \cos^{-1} 0 = \frac{\pi}{2}$$

$$\text{When } x = 1, t = \cos^{-1} 1 = 0$$

$$\therefore I = \int_{\frac{\pi}{2}}^0 t^2 \cdot (-\sin t) \cdot dt$$

$$= - \int_{\frac{\pi}{2}}^0 t^2 \sin t \cdot dt$$

$$= \int_0^{\frac{\pi}{2}} t^2 \sin t \cdot dt \quad \dots \left[ \because \int_a^b f(x) \cdot dx = - \int_b^a f(x) \cdot dx \right]$$

$$= \left[ t^2 \int \sin t \cdot dt \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left[ \frac{d}{dt} (t^2) \int \sin t \cdot dt \right] \cdot dt$$

$$= \left[ t^2 (\cos t) \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2t \cdot (-\cos t) \cdot dt$$

$$= \left[ -t^2 \cos t \right]_0^{\frac{\pi}{2}} + 2 \int_0^{\frac{\pi}{2}} t \cdot \cos t \cdot dt$$

$$= \left[ -\frac{\pi}{4} \cos \frac{\pi}{2} + 0 \right] + 2 \left\{ \left[ t \int \cos t \cdot dt \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left[ \frac{d}{dt} (t) \int \cos t \cdot dt \right] \cdot dt \right\}$$

$$\begin{aligned}
&= 0 + 2 \left\{ [t \sin t]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 1 \cdot \sin t \cdot dt \right\} \dots \left[ \because \cos \frac{\pi}{2} = 0 \right] \\
&= 2[t \sin t]_0^{\frac{\pi}{2}} - 2[(-\cos t)]_0^{\frac{\pi}{2}} \\
&= 2 \left[ \frac{\pi}{2} \sin \frac{\pi}{2} - 0 \right] - 2 \left[ -\cos \frac{\pi}{2} + \cos 0 \right] \\
&= 2 \left[ \frac{\pi}{2} \times 1 \right] - 2[-0 + 1] \\
&= \pi - 2.
\end{aligned}$$

Miscellaneous Exercise 4 | Q 2.07 | Page 176

Evaluate the following :  $\int_{-1}^1 \frac{1+x^3}{9-x^2} \cdot dx$

**SOLUTION**

$$\begin{aligned}
\text{Let } I &= \int_{-1}^1 \frac{1+x^3}{9-x^2} \cdot dx \\
&= \int_{-1}^1 \left[ \frac{1}{9-x^2} + \frac{x^3}{9-x^2} \right] \cdot dx \\
&= \int_{-1}^1 \frac{1}{9-x^2} \cdot dx + \int_{-1}^1 \frac{x^3}{9-x^2} \cdot dx \\
\therefore I &= I_1 + I_2 \quad \dots(1)
\end{aligned}$$

$$\begin{aligned}
I_1 &= \int_{-1}^1 \frac{1}{3^2 - x^2} \cdot dx \\
&= \frac{1}{2 \times 3} \left[ \log \left| \frac{3+x}{3-x} \right| \right]_{-1}^1
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{6} \left[ \log \left( \frac{4}{2} \right) - \log \left( \frac{2}{4} \right) \right] \\
&= \frac{1}{6} \left[ \log \left( \frac{2}{\frac{1}{2}} \right) \right] \\
&= \frac{1}{6} \log 4 \\
&= \frac{1}{6} \log 2^2 \\
&= \frac{1}{6} \times 2 \log 2 \\
&= \frac{1}{3} \log 2 \quad \dots(2)
\end{aligned}$$

$$I_2 = \int_{-1}^1 \frac{x^3}{9 - x^2} \cdot dx$$

$$\text{Let } f(x) = \frac{x^3}{9 - x^2}$$

$$\therefore f(-x) = \frac{(-x)^3}{9 - (-x)^2}$$

$$= \frac{(-x)^3}{9 - x^2}$$

$$= -f(x)$$

$\therefore f$  is an odd function.

$$\therefore \int_{-1}^1 f(x) \cdot dx = 0$$

$$\therefore I_2 = \int_{-1}^1 \frac{x^3}{9-x^2} \cdot dx = 0 \quad \dots(3)$$

From (1),(2) and (3), we get

$$\begin{aligned} I &= \frac{1}{3} \log 2 + 0 \\ &= \frac{1}{3} \log 2. \end{aligned}$$

#### Miscellaneous Exercise 4 | Q 2.08 | Page 176

Evaluate the following :  $\int_0^{\pi} x \cdot \sin x \cdot \cos^4 x \cdot dx$

#### **SOLUTION**

$$\text{Let } I = \int_0^{\pi} x \cdot \sin x \cdot \cos^4 x \cdot dx \quad \dots(1)$$

We use the property,  $\int_0^a f(x) \cdot dx = \int_0^a f(a-x) \cdot dx$

Here  $a = \pi$ .

Hence changing  $x$  by  $\pi - x$ , we get

$$\begin{aligned} I &= \int_0^{\pi} (\pi - x) \cdot \sin(\pi - x) \cdot [\cos(\pi - x)]^4 \cdot dx \\ &= \int_0^{\pi} (\pi - x) \cdot \sin x \cdot \cos^4 x \cdot dx \quad \dots(2) \end{aligned}$$

Adding(1) and (2), we get

$$\begin{aligned} 2I &= \int_0^{\pi} x \cdot \sin x \cdot \cos^4 x \cdot dx + \int_0^{\pi} (\pi - x) \cdot \sin x \cdot \cos^4 x \cdot dx \\ &= \int_0^{\pi} (x + \pi - x) \cdot \sin x \cdot \cos^4 x \cdot dx \end{aligned}$$

$$= \pi \int_0^{\pi} \sin x \cdot \cos^4 x \cdot dx$$

$$\therefore I = \frac{\pi}{2} \int_0^{\pi} \cos^4 x \cdot \sin x \cdot dx$$

Put  $\cos = t$

$$\therefore -\sin x \cdot dx = dt$$

$$\therefore \sin x \cdot dx = -dt$$

$$\text{When } x = 0, t = \cos 0 = 1$$

$$\text{When } x = \pi \cos \pi = -1$$

$$\therefore I = \frac{\pi}{2} \int_1^{-1} t^4 (-dt)$$

$$= -\frac{\pi}{2} \int_1^{-1} t^4 \cdot dt$$

$$= -\frac{\pi}{2} \left[ \frac{t^5}{5} \right]_1^{-1}$$

$$= -\frac{\pi}{10} [t^5]_1^{-1}$$

$$= -\frac{\pi}{10} [(-1)^5 - (1)^5]$$

$$= -\frac{\pi}{10} (-1 - 1)$$

$$= \frac{2\pi}{10}$$

$$= \frac{\pi}{5}$$

Miscellaneous Exercise 4 | Q 2.09 | Page 176

Evaluate the following :  $\int_0^{\pi} \frac{x}{1 + \sin^2 x} \cdot dx$



**SOLUTION**

$$\text{Let } I = \int_0^{\pi} \frac{x}{1 + \sin^2 x} \cdot dx \quad \dots(1)$$

$$\text{We use the property, } \int_a^b f(x) \cdot dx = \int_0^a f(a-x) \cdot dx$$

Here  $a = \pi$ .

Hence in I, changing  $x$  to  $\pi - x$ , we get

$$\begin{aligned} I &= \int_0^{\pi} \frac{\pi - x}{1 + \sin^2(\pi - x)} \cdot dx \\ &= \int_0^{\pi} \frac{\pi - x}{1 + \sin^2 x} \cdot dx \\ &= \int_0^{\pi} \frac{\pi}{1 + \sin^2 x} \cdot dx \\ &= - \int_0^{\pi} \frac{x}{1 + \sin^2 x} \cdot dx \\ &= \int_0^{\pi} \frac{\pi}{1 + \sin^2 x} \cdot dx - I \quad \dots[\text{By (1)}] \\ \therefore 2I &= \pi \int_0^{\pi} \frac{1}{1 + \sin^2 x} \cdot dx \end{aligned}$$

Dividing numerator and denominator by  $\cos^2 x$ , we get

$$\begin{aligned} 2I &= \pi \int_0^{\pi} \frac{\sec^2 x}{\sec^2 x + \tan^2 x} \cdot dx \\ &= \pi \int_0^{\pi} \frac{\sec^2 x}{1 + 2 \tan^2 x} \cdot dx \end{aligned}$$

Put  $\tan x = t$

$$\therefore \sec^2 x \cdot dx = dt$$

When  $x = \pi$ ,  $t = \tan \pi = 0$

When  $x = 0$ ,  $t = \tan 0 = 0$

$$\therefore 2I = \pi \int_0^\pi \frac{dt}{1+t^2} = 0$$

$$\therefore I = 0. \quad \dots \left[ \because \int_a^a f(x) \cdot dx = 0 \right]$$

**Miscellaneous Exercise 4 | Q 3.01 | Page 176**

Evaluate the following :  $\int_0^1 \left( \frac{1}{1+x^2} \right) \sin^{-1} \left( \frac{2x}{1+x^2} \right) \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_0^1 \left( \frac{1}{1+x^2} \right) \sin^{-1} \left( \frac{2x}{1+x^2} \right) \cdot dx$$

Put  $x = \tan t$ , i.e.  $t = \tan^{-1} x$

$$\therefore dx = \sec^2 t \, dt$$

$$\text{When } x = 1, t = \tan^{-1} 1 = \frac{\pi}{4}$$

$$\text{When } x = 0, t = \tan^{-1} 0 = 0$$

$$\begin{aligned} \therefore I &= \int_0^{\frac{\pi}{4}} \left( \frac{1}{1+\tan^2 t} \right) \sin^{-1} \left( \frac{2 \tan t}{1+\tan^2 t} \right) \sec^2 t \cdot dt \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2 t} \sin^{-1}(\sin 2t) \sec^2 t \cdot dt \\ &= \int_0^{\frac{\pi}{4}} 2t \cdot dt \end{aligned}$$

$$= 2 \int_0^{\frac{\pi}{4}} t \cdot dt$$

$$= 2 \left[ \frac{t^2}{2} \right]_0^{\frac{\pi}{4}}$$

$$= 2 \left[ \frac{\pi}{32} - 0 \right]$$

$$= \frac{\pi^2}{16}.$$

Miscellaneous Exercise 4 | Q 3.02 | Page 176

Evaluate the following :  $\int_0^{\frac{\pi}{2}} \frac{1}{6 - \cos x} \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{1}{6 - \cos x} \cdot dx$$

$$\text{Put } \tan\left(\frac{x}{2}\right) = t$$

$$\therefore x = 2 \tan^{-1} t$$

$$\therefore dx = \frac{2dt}{1+t^2}$$

and

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\text{When } x = \frac{\pi}{2}, t = \tan\left(\frac{\pi}{2}\right) = 1$$

$$\text{When } x = 0, t = \tan 0 = 0$$

$$\begin{aligned}
 \therefore I &= \frac{\frac{2dt}{1+t^2}}{6 - \cos\left(\frac{1-t^2}{1+t^2}\right)} \\
 &= \int_0^1 \frac{2dt}{6(1+t^2) + 1(1-t^2)} \\
 &= 2 \int_0^1 \frac{1}{t^2 + 7} \cdot dt \\
 &= 2 \left[ \frac{1}{35} \tan^{-1} \frac{t}{5} \right]_0^1 \\
 &= 2 \left[ \frac{1}{35} \tan^{-1} \frac{1}{5} - \frac{1}{35} \tan^{-1} 0 \right] \\
 &= \frac{2}{35} \tan^{-1} \frac{1}{5} - \frac{2}{35} \times 0 \\
 &= \frac{2}{35} \tan^{-1} \frac{1}{5}.
 \end{aligned}$$

Miscellaneous Exercise 4 | Q 3.03 | Page 176

Evaluate the following :  $\int_0^a \frac{1}{a^2 + ax - x^2} \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_0^a \frac{1}{a^2 + ax - x^2} \cdot dx$$

$$a^2 + ax - x^2 = a^2 - \left(x^2 - ax + \frac{a^2}{4}\right) + \frac{a^2}{4}$$

$$= \frac{5a^2}{4} - \left(x - \frac{a}{2}\right)^2$$

$$= \left(\frac{\sqrt{5}a}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2$$

$$\therefore I = \int_0^a \frac{dx}{\left(\frac{\sqrt{5}a}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2}$$

$$= \frac{1}{\frac{2 \times \sqrt{5}a}{2}} \cdot \left[ \log \left| \frac{\frac{\sqrt{5}a}{2} + x - \frac{a}{2}}{\frac{\sqrt{5}a}{2} - x + \frac{a}{2}} \right| \right]_0^a$$

$$= \frac{1}{\sqrt{5}a} \left[ \log \left| \frac{\frac{\sqrt{5}a}{2} + a - \frac{a}{2}}{\frac{\sqrt{5}a}{2} - a + \frac{a}{2}} \right| - \log \left| \frac{\frac{\sqrt{5}a}{2} - \frac{a}{2}}{\frac{\sqrt{5}a}{2} + \frac{a}{2}} \right| \right]$$

$$= \frac{1}{\sqrt{5}a} \left[ \log \left| \frac{\frac{\sqrt{5}}{2} + \frac{1}{2}}{\frac{\sqrt{5}}{2} - \frac{1}{2}} \right| - \log \left| \frac{\frac{\sqrt{5}}{2} - \frac{1}{2}}{\frac{\sqrt{5}}{2} + \frac{1}{2}} \right| \right]$$

$$= \frac{1}{\sqrt{5}a} \left[ \log \left| \left( \frac{\sqrt{5} + 1}{\sqrt{5} - 1} \right) \right| - \log \left| \left( \frac{\sqrt{5} - 1}{\sqrt{5} + 1} \right) \right| \right]$$

$$= \frac{1}{\sqrt{5}a} \log \left| \frac{\sqrt{5} + 1}{\sqrt{5} - 1} \times \frac{\sqrt{5} + 1}{\sqrt{5} - 1} \right|$$

$$\begin{aligned}
&= \frac{1}{\sqrt{5}a} \log \left[ \left( \frac{\sqrt{5}+1}{\sqrt{5}-1} \right)^2 \right] \\
&= \frac{1}{\sqrt{5}a} \log \left| \frac{5+1+2\sqrt{5}}{5+1-2\sqrt{5}} \right| \\
&= \frac{1}{\sqrt{5}a} \log \frac{6+2\sqrt{5}}{6-2\sqrt{5}} \\
&= \frac{1}{\sqrt{5}a} \log \left| \frac{6+2\sqrt{5}}{6-2\sqrt{5}} \times \frac{6+2\sqrt{5}}{6+2\sqrt{5}} \right| \\
&= \frac{1}{\sqrt{5}a} \log \left| \frac{36+20+24\sqrt{5}}{36-20} \right| \\
&= \frac{1}{\sqrt{5}a} \log \left| \frac{56+24\sqrt{5}}{16} \right| \\
&= \frac{1}{\sqrt{5}a} \log \left| \frac{7+3\sqrt{5}}{2} \right|.
\end{aligned}$$

Miscellaneous Exercise 4 | Q 3.04 | Page 176

Evaluate the following :  $\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\sin x}{\sin x + \cos x} \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\sin x}{\sin x + \cos x} \cdot dx \quad \dots(1)$$

We use the property,  $\int_a^b f(x) \cdot dx = \int_a^b f(a + b - x) \cdot dx$ .

$$\text{Here } a = \frac{\pi}{5}, b = \frac{3\pi}{10}.$$

Hence changing  $x$  by  $\frac{\pi}{5} + \frac{3\pi}{10} - x$ , we get,

$$\begin{aligned} I &= \int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\sin\left(\frac{\pi}{5} + \frac{3\pi}{10} - x\right)}{\sin\left(\frac{\pi}{5} + \frac{\pi}{10} - x\right) + \cos\left(\frac{\pi}{5} + \frac{3\pi}{10} - x\right)} \cdot dx \\ &= \int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} \cdot dx \\ &= \int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\cos x}{\cos x + \sin x} \cdot dx \quad \dots(2) \end{aligned}$$

Adding (1) and (2), we get,

$$\begin{aligned} 2I &= \int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\sin x}{\sin x + \cos x} \cdot dx + \int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\cos x}{\cos x + \sin x} \cdot dx \\ &= \int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\sin x + \cos x}{\sin x + \cos x} \cdot dx \\ &= \int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} 1 \cdot dx = [x]_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \\ &= \frac{3\pi}{10} - \frac{\pi}{5} \end{aligned}$$

$$= \frac{\pi}{10}$$

$$\therefore I = \frac{\pi}{20}.$$

Miscellaneous Exercise 4 | Q 3.05 | Page 176

Evaluate the following :  $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) \cdot dx$$

Put  $x = \tan t$ , i.e.  $t = \tan^{-1}x$

$$\therefore dx = \sec^2 t \cdot dt$$

When  $x = 0$ ,  $t = \tan^{-1} 0 = 0$

When  $x = 1$ ,  $t = \tan^{-1} 1 = \frac{\pi}{4}$

$$\therefore I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2 \tan t}{1 + \tan^2 t}\right) \sec^2 t \cdot dt$$

$$= \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2t) \sec^2 t \cdot dt$$

$$= \int_0^{\frac{\pi}{4}} 2t \sec^2 t \cdot dt$$

$$= \left[ 2t \int \sec^2 t \cdot dt \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \left[ \frac{d}{dx}(2t) \int \sec^2 t \cdot dt \right]$$

$$= [2t \tan t]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2 \tan t \cdot dt$$



$$\begin{aligned}
&= \left[ 2 \cdot \frac{\pi}{4} \tan \frac{\pi}{4} - 0 \right] - 2 \log(\sec t) \Big|_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{2} - 2 \left[ \log \left( \sec \frac{\pi}{4} \right) - \log(\sec 0) \right] \\
&= \frac{\pi}{2} - 2 \left[ \log \sqrt{2} - \log 1 \right] \\
&= \frac{\pi}{2} - 2 \left[ \frac{1}{2} \log 2 - 0 \right] \\
&= \frac{\pi}{2} - \log 2.
\end{aligned}$$

Miscellaneous Exercise 4 | Q 3.06 | Page 176

Evaluate the following :  $\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{1 + \cos 2x + \sin 2x} \cdot dx$

**SOLUTION**

$$\begin{aligned}
\text{Let } I &= \int_0^{\frac{\pi}{4}} \frac{\cos 2x}{1 + \cos 2x + \sin 2x} \cdot dx \\
&= \int_0^{\frac{\pi}{4}} \frac{\cos^2 x - \sin^2 x}{2 \cos^2 x + 2 \sin x \cos x} \cdot dx \\
&= \int_0^{\frac{\pi}{4}} \frac{(\cos x - \sin x)(\cos x + \sin x)}{2 \cos x (\cos x + \sin x)} \cdot dx \\
&= \int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{2 \cos x} \cdot dx \\
&= \frac{1}{2} \int_0^{\frac{\pi}{4}} \left[ \frac{\cos x}{\cos x} - \frac{\sin x}{\cos x} \right] \cdot dx \\
&= \frac{1}{2} \left[ \int_0^{\frac{\pi}{4}} 1 \cdot dx - \int_0^{\frac{\pi}{4}} \tan x \cdot dx \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left\{ [x]_0^{\frac{\pi}{4}} - [\log(\sec x)]_0^{\frac{\pi}{4}} \right\} \\
&= \frac{1}{2} \left[ \left( \frac{\pi}{4} - 0 \right) - \left( \log \sec \frac{\pi}{4} - \log \sec 0 \right) \right] \\
&= \frac{1}{2} \left[ \frac{\pi}{4} - \log \sqrt{2} + \log 1 \right] \\
&= \frac{1}{2} \left[ \frac{\pi}{4} - \log \sqrt{2} \right]. \quad \dots [\because \log 1 = 0]
\end{aligned}$$

**Miscellaneous Exercise 4 | Q 3.07 | Page 176**

Evaluate the following :  $\int_0^{\frac{\pi}{2}} [2 \log(\sin x) - \log(\sin 2x)] \cdot dx$

**SOLUTION**

$$\begin{aligned}
\text{Let } I &= \int_0^{\frac{\pi}{2}} (2 \log \sin x - \log \sin 2x) \cdot dx \\
&= \int_0^{\frac{\pi}{2}} [2 \log \sin x - \log(2 \sin x \cos x)] \cdot dx \\
&= \int_0^{\frac{\pi}{2}} [2 \log \sin x - (\log 2 + \log \sin x + \log \cos x)] \cdot dx \\
&= \int_0^{\frac{\pi}{2}} (2 \log \sin x - \log 2 - \log \sin x - \log \cos x) \cdot dx \\
&= \int_0^{\frac{\pi}{2}} (\log \sin x - \log \cos x - \log 2) \cdot dx \\
&= \int_0^{\frac{\pi}{2}} \log \sin x \cdot dx - \int_0^{\frac{\pi}{2}} \log \cos x \cdot dx - \log 2 \int_0^{\frac{\pi}{2}} 1 \cdot dx \\
&= \int_0^{\frac{\pi}{2}} \log \left[ \sin \left( \frac{\pi}{2} - x \right) \right] \cdot dx - \int_0^{\frac{\pi}{2}} \log \cos x \cdot dx - \log 2 [x]_0^{\frac{\pi}{2}} \quad \dots \left[ \because \int_0^a f(x) \cdot dx = \int_0^a f(a-x) \cdot dx \right] \\
&= \int_0^{\frac{\pi}{2}} \log \cos x \cdot dx - \int_0^{\frac{\pi}{2}} \log \cos x \cdot dx - \log 2 \left[ \frac{\pi}{2} - 0 \right] \\
&= -\frac{\pi}{2} \log 2.
\end{aligned}$$

Evaluate the following :  $\int_0^{\pi} (\sin^{-1} x + \cos^{-1} x)^3 \sin^3 x \cdot dx$

**SOLUTION**

$$\text{Let } I = \int_0^{\pi} (\sin^{-1} x + \cos^{-1} x)^3 \sin^3 x \cdot dx$$

We know that,  $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

and

$$\sin 3x = 3 \sin x - 4 \sin^3 x$$

$$\therefore 4 \sin^3 x = 3 \sin x - \sin 3x$$

$$\therefore \sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$$

$$\begin{aligned} \therefore I &= \int_0^{\pi} \left( \frac{\pi}{2} \right)^3 \left[ \frac{3}{4} \sin x - \frac{1}{4} \sin 3x \right] \cdot dx \\ &= \frac{\pi^3}{8} \times \frac{3}{4} \int_0^{\pi} \sin x \cdot dx - \frac{\pi^3}{8} \times \frac{1}{4} \int_0^{\pi} \sin 3x \\ &= \frac{3\pi^3}{32} [-\cos \pi - (-\cos 0)] - \frac{\pi^3}{32} \left[ -\frac{\cos 3\pi}{3} - \left( -\frac{\cos 0}{3} \right) \right] \\ &= \frac{3\pi^3}{32} [1 + 1] - \frac{\pi^3}{32} \left[ \frac{1}{3} + \frac{1}{3} \right] \\ &= \frac{6\pi^3}{32} - \frac{2\pi^3}{96} \\ &= \frac{18\pi^3 - 2\pi^3}{96} \\ &= \frac{16\pi^3}{96} \end{aligned}$$

$$= \frac{\pi^3}{6}.$$

Miscellaneous Exercise 4 | Q 3.09 | Page 176

Evaluate the following :  $\int_0^4 \left[ \sqrt{x^2 + 2x + 3} \right]^{-1} \cdot dx$

**SOLUTION**

$$\begin{aligned} \text{Let } I &= \int_0^4 \left[ \sqrt{x^2 + 2x + 3} \right]^{-1} \cdot dx \\ &= \int_0^4 \frac{1}{\sqrt{x^2 + 2x + 1 + 2}} \cdot dx \\ &= \int_0^4 \frac{1}{\sqrt{(x+1)^2 + 2}} \cdot dx \\ &= \left[ \log \left[ x + 1 + \sqrt{(x+1)^2 + 2} \right] \right]_0^4 \\ &= \log \left[ 4 + 1 + \sqrt{5^2 + 2} \right] - \log \left[ 0 + 1 + \sqrt{1^2 + 2} \right] \\ &= \log \left( 5 + 3\sqrt{3} \right) - \log \left( 1 + \sqrt{3} \right) \\ &= \log \left( \frac{5 + 3\sqrt{3}}{1 + \sqrt{3}} \right). \end{aligned}$$

Miscellaneous Exercise 4 | Q 3.1 | Page 176

Evaluate the following :  $\int_{-2}^3 |x - 2| \cdot dx$

**SOLUTION**

$$|x - 2| = 2 - x, \text{ if } x < 2$$

$$= x - 2, \text{ if } x \geq 2$$

$$\therefore \int_{-2}^3 |x - 2| \cdot dx = \int_{-2}^2 |x - 2| \cdot dx + \int_2^3 |x - 2| \cdot dx$$

$$= \int_{-2}^2 (2 - x) \cdot dx + \int_2^3 (x - 2) \cdot dx$$

$$= \left[ \frac{(2 - x)^2}{(-2)} \right]_{-2}^2 + \left[ \frac{(x - 2)^2}{2} \right]_2^3$$

$$= \left[ 0 - \frac{(4)^2}{(-2)^2} \right] + \left[ \frac{1^2}{2} - \frac{0^2}{2} \right]$$

$$= 8 + \frac{1}{2}$$

$$= \frac{17}{2}.$$

Miscellaneous Exercise 4 | Q 4.1 | Page 177

Evaluate the following : if  $\int_a^a \sqrt{x} \cdot dx = 2a \int_0^{\frac{\pi}{2}} \sin^3 x \cdot dx$ , find the value of  $\int_a^{a+1} x \cdot dx$

**SOLUTION**

It is given that

$$\int_a^a \sqrt{x} \cdot dx = 2a \int_0^{\frac{\pi}{2}} \sin^3 x \cdot dx$$

$$\therefore \left[ \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^a = 2a \cdot \frac{2}{3} \dots [\text{Using Reduction Formula}]$$

$$\therefore \left[ \frac{2a^{\frac{3}{2}}}{3} - 0 \right] = \frac{4a}{3}$$

$$\therefore \frac{2a\sqrt{a}}{3} = \frac{4a}{3}$$

$$\therefore 2a(\sqrt{a} - 2) = 0$$

$$\therefore a = 0 \text{ or } \sqrt{a} = 2$$

$$\text{i.e. } a = 0 \text{ or } a = 4$$

$$\text{When } a = 0, \int_a^{a+1} x \cdot dx = \int_0^1 x \cdot dx$$

$$= \left[ \frac{x^2}{2} \right]_0^1$$

$$= \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

$$\text{When } a = 4, \int_a^{a+1} x \cdot dx = \int_4^5 x \cdot dx$$

$$= \left[ \frac{x^2}{2} \right]_4^5$$

$$= \frac{25}{2} - \frac{16}{2}$$

$$= \frac{9}{2}$$

Miscellaneous Exercise 4 | Q 4.2 | Page 177

Evaluate the following : If  $\int_0^k \frac{1}{2+8x^2} \cdot dx = \frac{\pi}{16}$ , find k

**SOLUTION**

$$\begin{aligned}\text{Let } I &= \int_0^k \frac{1}{2+8x^2} \cdot dx \\&= \frac{1}{8} \int_0^k \frac{1}{x^2 + \left(\frac{1}{2}\right)^2} \cdot dx \\&= \frac{1}{8} \times \frac{1}{\left(\frac{1}{2}\right)} \left[ \tan^{-1} \left( \frac{x}{\left(\frac{1}{2}\right)} \right) \right]_0^k \\&= \frac{1}{4} [\tan^{-1} 2x]_0^k \\&= \frac{1}{4} [\tan^{-1} 2k - \tan^{-1} 0] \\&= \frac{1}{4} \tan^{-1} 2k \\ \therefore I &= \frac{\pi}{16} \text{ gives } \frac{1}{4} \tan^{-1} 2k = \frac{\pi}{16} \\ \therefore \tan^{-1} 2k &= \frac{\pi}{4} \\ \therefore 2k &= \tan \frac{\pi}{4} = 1 \\ \therefore k &= \frac{1}{2}.\end{aligned}$$

Miscellaneous Exercise 4 | Q 4.3 | Page 177

Evaluate the following : If  $f(x) = a + bx + cx^2$ , show that  $\int_0^1 f(x) \cdot dx = \left( \frac{1}{6} \left[ f(0) + 4f\left(\frac{1}{2}\right) + f(1) \right] \right)$

**SOLUTION**

$$\begin{aligned}
 \int_0^1 f(x) \cdot dx &= \int_0^1 (a + bx + cx^2) \cdot dx \\
 &= a \int_0^1 1 \cdot dx + b \int_0^1 x \cdot dx + c \int_0^1 x^2 \cdot dx \\
 &= \left[ ax + \frac{bx^2}{2} + \frac{cx^3}{3} \right]_0^1 \\
 &= a + \frac{b}{2} + \frac{c}{3} \quad \dots(1)
 \end{aligned}$$

Now,  $f(0) = a + b(0) + c(0)^2 = a$

$$f\left(\frac{1}{2}\right) = a + b\left(\frac{1}{2}\right) + c\left(\frac{1}{2}\right)^2 = a + \frac{b}{2} + \frac{c}{4}$$

and

$$f(1) = a + b(1) + c(1)^2 = a + b + c$$

$$\begin{aligned}
 \therefore \frac{1}{6} \left[ f(0) + 4f\left(\frac{1}{2}\right) + f(1) \right] \\
 = \frac{1}{6} \left[ a + 4\left(a + \frac{b}{2} + \frac{c}{4}\right) + (a + b + c) \right]
 \end{aligned}$$

$$= \frac{1}{6} [a + 4a + 2b + c + a + b + c]$$

$$= \frac{1}{6} [6a + 3b + 2c]$$

$$= a + \frac{b}{2} + \frac{c}{3} \quad \dots(2)$$

$\therefore$  from (1) and (2),

$$\int_0^1 f(x) \cdot dx = \frac{1}{6} \left[ f(0) + 4f\left(\frac{1}{2}\right) + f(1) \right].$$